Second Edition

Vibrations

Balakumar Balachandran

Edward B. Magrab

VIBRATIONS

SECOND EDITION

Balakumar Balachandran | Edward B. Magrab

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Preface

Vibration is a classical subject whose principles have been known and studied for many centuries and presented in many books. Over the years, the use of these principles to understand and design systems has seen considerable growth in the diversity of systems that are designed with vibrations in mind: mechanical, aerospace, electromechanical and microelectromechanical devices and systems, biomechanical and biomedical systems, ships and submarines, and civil structures. As the performance envelope of an engineered system is pushed to higher limits, nonlinear effects also have to be taken into account.

This book has been written to enable the use of vibration principles in a broad spectrum of applications and to meet the wide range of challenges faced by system analysts and designers. To this end, the authors have the following goals: a) to provide an introduction to the subject of vibrations for undergraduate students in engineering and the physical sciences, b) to present vibration principles in a general context and to illustrate the use of these principles through carefully chosen examples from different disciplines, c) to use a balanced approach that integrates principles of linear and nonlinear vibrations with modeling, analysis, prediction, and measurement so that physical understanding of the vibratory phenomena and their relevance for engineering design can be emphasized, and d) to deduce design guidelines that are applicable to a wide range of vibratory systems.

In writing this book, the authors have used the following guidelines. The material presented should have, to the extent possible, a physical relevance to justify its introduction and development. The examples should be relevant and wide ranging, and they should be drawn from different areas, such as biomechanics, electronic circuit boards and components, machines, machining (cutting) processes, microelectromechanical devices, and structures. There should be a natural integration and progression between linear and nonlinear systems, between the time domain and the frequency domain, among the responses of systems to harmonic and transient excitations, and between discrete and continuous system models. There should be a minimum emphasis placed on the discussion of numerical methods and procedures, per se, and instead, advantage should be taken of tools such as MATLAB for generating the numerical solutions and complementing analytical solutions. The algorithms for generating numerical solutions should be presented external to the chapters, as they tend to break the flow of the material being presented. (The MATLAB algorithms used to construct and generate all solutions can be found at the publisher's web site for this book.) Further advantage should be taken of tools such as MATLAB in concert with analysis, so that linear systems can be extended to include nonlinear elements. Finally, there should be a natural and integrated interplay and presentation between analysis, modeling, measurement, prediction, and design so that a reader does not develop artificial distinctions among them.

Many parts of this book have been used for classroom instruction in a vibrations course offered at the junior level at the University of Maryland. Typically, students in this course have had a sophomore-level course on dynamics and a course on ordinary differential equations that includes Laplace transforms. Beyond that, some fundamental material on complex numbers (Appendix F) and linear algebra (Appendix E) is introduced at the appropriate places in the course. Regarding obtaining the solution for response, our preference in most instances is to obtain the solution by using Laplace transforms. A primary motivation for using the Laplace transform approach is that it is used in the study of control systems, and it can be used with ease to show the duality between the time domain and the frequency domain. However, other means to solve for the response are also presented in Appendix D.

This book has the following features. Both Newton's laws and Lagrange's equations are used to develop models of systems. Since an important part of this development requires kinematics, kinematics is reviewed in Chapter 1. We use Laplace transforms to develop analytical solutions for linear vibratory systems and, from the Laplace domain, extend these results to the frequency domain. The responses of these systems are discussed in both the time and frequency domains to emphasize their duality. Notions of transfer functions and frequency-response functions also are used throughout the book to help the reader develop a comprehensive picture of vibratory systems. We have introduced design for vibration (DFV) guidelines that are based on vibration principles developed throughout the book. The guidelines appear at the appropriate places in each chapter. These design guidelines serve the additional function of summarizing the preceding material by encapsulating the most important elements as they relate to some aspect of vibration design. Many examples are included from the area of microelectromechanical systems throughout the book to provide a physical context for the application of principles of vibrations at "small" length scales. In addition, there are several examples of vibratory models from biomechanics. Throughout the book, extensive use has been made of MATLAB, and in doing so, we have been able to include a fair amount of new numerical results, which were not accessible or not easily accessible to analysis previously. These results reveal many interesting phenomena, which the authors believe help expand our understanding of vibrations.

The book is organized into nine chapters, with the topics covered ranging from pendulum systems and spring-mass-damper prototypes to beams. In mechanics, the subject of vibrations is considered a subset of dynamics, in which one is concerned with the motions of bodies subjected to forces and moments. For much of the material covered in this book, a background in dynamics on the plane is sufficient. In the introductory chapter (Chapter 1), a summary is provided of concepts such as degrees of freedom and principles such as Newton's linear momentum principle and Euler's angular momentum principle.

In the second chapter, the elements that are used to construct a vibratory system model are introduced and discussed. The notion of equivalent spring stiffness is presented in different physical contexts. Different damping models that can be used in modeling vibratory systems also are presented in this chapter. A section on design for vibration has been added to this edition. In Chapter 3, the derivation of the equation governing a single degree-of-freedom vibratory system is addressed. For this purpose, principles of linear momentum balance and angular momentum balance and Lagrange's equations are used. Notions such as natural frequency and damping factor also are introduced here. Linearization of nonlinear systems also is explained in this chapter. In the fourth chapter, responses to different initial conditions, including impact, are examined. Responses of systems with linear springs and nonlinear springs also are compared here. Free-oscillation characteristics of systems with nonlinear damping also are studied. The notion of stability is briefly addressed, and the reader is introduced to the important phenomenon of machine-tool chatter.

In Chapter 5, the responses of single degree-of-freedom systems subjected to periodic excitations are considered. The notions of resonance, frequency-response functions, and transfer functions are discussed in detail. The responses of linear and nonlinear vibratory systems subjected to harmonic excitations also are examined. The Fourier transform is introduced, and considerable attention is paid to relating the information in the time domain to the frequency domain and vice versa. For different excitations, sensitivity of frequency-response functions with respect to the system parameters also is examined for design purposes. Accelerometer design is discussed, and the notion of equivalent damping is presented. This edition of the book also includes a section on alternative forms of the frequency-response function. In Chapter 6, the responses of single degree-of-freedom systems to different types of external transient excitations are addressed and analyzed in terms of their frequency spectra relative to the amplitude-response function of the system. The notion of a spectral energy is used to study vibratory responses, and a section on impact testing has been added.

Multiple degree-of-freedom systems are treated in Chapters 7 and 8 leading up to systems with an infinite number of degrees of freedom in Chapter 9. In Chapter 7, the derivation of governing equations of motion of a system with multiple degrees of freedom is addressed by using the principles of linear momentum balance and angular momentum balance and Lagrange's equations. The natural frequencies and mode shapes of undamped systems also are studied in this chapter, and the notion of a vibratory mode is explained. Linearization of nonlinear multiple degree-of-freedom systems and systems with gyroscopic forces also are treated in this chapter. Stability notions discussed in Chapter 4 for a single degree-of-freedom system are extended to multiple degree-of-freedom systems, and conservation of energy and momentum are studied. In this edition, a section on vibrations of rotating shafts on flexible supports has been added.

In Chapter 8, different approaches that can be used to obtain the response of a multiple degree-of-freedom system are presented. These approaches include the direct approach, the normal-mode approach, the Laplace transform approach, and the one based on state-space formulation. Explicit solution forms for responses of multiple degree-of-freedom systems are obtained and used to arrive at the response to initial conditions and different types of forcing. The importance of the normal-mode approach to carry out modal analysis of vibratory systems with special damping properties is addressed in this chapter. The state-space formulation is used to show how vibratory systems with arbitrary forms of damping can be treated. The notion of resonance in a multiple degree-of-freedom system is addressed here. Notions of frequency-response functions and transfer functions, which were introduced in Chapter 5 for single degree-of-freedom systems, are revisited, and the relevance of these notions for system identification and design of vibration absorbers, mechanical filters, and vibration isolation systems is brought forth in Chapter 8. The vibration-absorber material includes the traditional treatment of linear vibration absorbers and a brief introduction to the design of nonlinear vibration absorbers, which include a bar-slider system, a pendulum absorber, and a particle-impact damper. Tools based on optimization techniques are also introduced for tailoring vibration absorbers and vibration isolation systems.

In Chapter 9, the subject of beam vibrations is treated at length as a representative example of vibrations of systems with an infinite number of degrees of freedom. The derivation of governing equations of motion for isotropic beams is addressed and both free and forced oscillations of beams are studied for an extensive number of boundary conditions and interior and exterior attachments. In particular, considerable attention is paid to free-oscillation characteristics such as mode shapes, and effects of axial forces, elastic foundation, and beam geometry on these characteristics. A large number of numerical results that do not appear elsewhere are included here. In Chapter 9, the power of the Laplace transform approach to solve the beam response for complex boundary conditions is illustrated. Furthermore, this edition also includes an appendix on the natural frequencies and mode shapes associated with the free oscillations of strings, bars, and shafts, each for various combinations of boundary conditions including an attached mass and an attached spring. Also presented in the appendix are results that can be used to determine when the systems can be modeled as single degree-of-freedom systems.

This edition of the book includes several aids aimed at facilitating the reader with the material. In the introduction of each chapter, a discussion is provided on what specifically will be covered in that chapter. The examples have been chosen so that they are of different levels of complexity, cover a wide range of vibration topics and, in most cases, have practical applications to real-world problems. The exercises have been reorganized to correlate with the most appropriate section of the text. A glossary has been added to list in one place the definitions of the major terms used in the book. Finally, this edition of the book includes seven appendices on the following: i) Laplace transform pairs, ii) Fourier series, iii) notion of the decibel, iv) complex numbers and variables, v) linear algebra, vi) solution methods to second-order ordinary differential equations, and vii) natural frequencies and mode shapes of bar, shafts, and strings.

In terms of how this book can be used for a semester-long undergraduate course, our experience at the University of Maryland has been the following. In a course format with about 28 seventy-five minute lectures, we have been able to cover the following material: Chapter 1; Chapter 2 excluding Section 2.5; Chapter 3; Sections 4.1 to 4.3 of Chapter 4; Chapter 5 excluding Sections 5.3.3, 5.8, 5.9, and 5.10; Sections 6.1 to 6.3 of Chapter 6; Sections 7.1 to 7.3 of Chapter 7 excluding Sections 7.2.3, 7.3.3, and 7.3.4; and Sections 8.1, 8.2, 8.4, 8.5, and 8.6.1 of Chapter 8. We also have used this book in a format with 28 fifty-minute lectures and 14 ninety-minute-long studio sessions for an undergraduate course. In courses with lecture sessions and studio sessions, the studio sessions can include MATLAB studios and physical experiments, and in this format, one may be able to address material from Sections 2.5, 4.4, 4.5, 5.10, 7.2, and 8.6. Of course, there are sections such as Section 4.2 of Chapter 4, which may be too long to be covered in its entirety. In sections such as these, it is important to strike a balance through a combination of reading assignments and classroom instruction. Our experience is that a careful choice of periodic reading assignments can help the instructor cover a considerable amount of material, if desired. We also encourage an instructor to take advantage of the large number of examples provided in this book. Chapter 9 is not covered during the classroom lectures, but students are encouraged to explore material in this chapter through the project component of the course if appropriate. It is also conceivable that Chapters 6, 7, 8, and 9 can form the core of a graduate course on vibrations.

We express our sincere thanks to our former students for their spirited participation with regard to earlier versions of this book and for providing feedback; to the reviewers of this manuscript for their constructive suggestions; our colleagues Professor Bruce Berger for his careful reading of Chapter 1, Professor Amr Baz for suggesting material and examples for inclusion, Professor Donald DeVoe for pointing us to some of the literature on microelectromechanical systems, and Dr. Henry Haslach for reading and commenting on parts of Chapter 9; Professor Miao Yu for using this book in the classroom and providing feedback, especially with regard to Chapter 5; Professor Jae-Eun Oh of Hanyang University, South Korea, for spending a generous amount of time in reading the early versions of Chapters 1 through 6 and providing feedback for the material as well as suggestions for the exercises and their solutions; and Professor Sergio Preidikman of University of Córdoba, Argentina, for using this book in the classroom, providing feedback to enhance the book, as well as for pointing out many typographical errors in the first edition. We would like to thank Professor Bingen Yang of the University of Southern California, Professor Robert G. Parker of the Ohio State University, and Professor Kon-Well Wang of the Pennsylvania State University for their helpful comments and suggestions during the preparation of the first edition of this book. In addition, we would also like to thank Professor Leonard Louis Koss of Monash University, Australia; Professor Robert G. Langlois of Carleton University, Canada; Professor Nicholas Haritos of the University of Melbourne, Australia; and Professor Chuck van Karsen of the Michigan Technological University for their constructive reviews while preparing the second edition of this book. We are also thankful to Mr. William Stenquist of Cengage-Learning for tirelessly supporting and encouraging the first edition of this book and Mr. Christopher Carson of Cengage Learning for his support of the second edition. Last, but not least, we are grateful to our families for being tremendously supportive and understanding of us throughout this time-consuming undertaking.

B. Balachandran
E. B. Magrab

College Park, MD

Table of Examples

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many of the mathematical developments that are commonly taught in a vibrations course can be traced back to the 1800s and before. However, since then, the use of these principles to understand and design systems has seen considerable growth in the diversity of systems that are designed with vibrations in mind: mechanical, electromechanical and microelectromechanical devices and systems, biomechanical and biomedical systems, ships and submarines, and civil structures.

In this chapter, we shall show how to:

- Determine the displacement, velocity, and acceleration of a mass element.
- Determine the number of degrees of freedom.
- Determine the kinetic energy and the work of a system.

1.2 PRELIMINARIES FROM DYNAMICS

Dynamics can be thought as having two parts, one being kinematics and the other being kinetics. While kinematics deals with the mathematical description of motion, kinetics deals with the physical laws that govern a motion. Here, first, particle kinematics and rigid-body kinematics are reviewed. Then, the notions of generalized coordinates and degrees of freedom are discussed. Following that, particle dynamics and rigid-body dynamics are addressed and the principles of linear momentum and angular momentum are presented. Finally, work and energy are discussed.

1.2.1 Kinematics of Particles and Rigid Bodies

Particle Kinematics

In Figure 1.1, a particle in free space is shown. In order to study the motions of this particle, a reference frame R and a set of unit vectors[2] $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ fixed

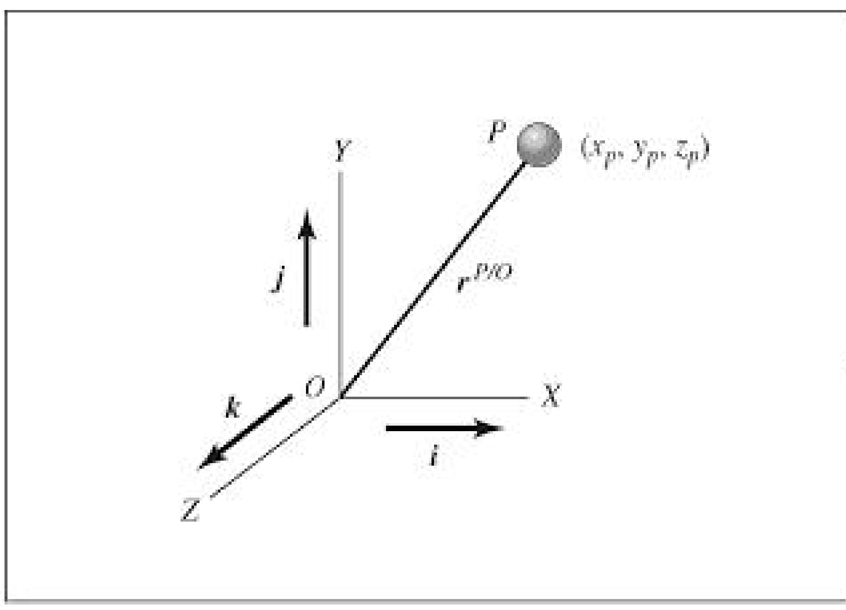

FIGURE 1.1
Particle kinematics. R is reference frame in which the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are fixed.

[2]As a convention throughout the book, bold and italicized letters represent vectors.

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EXAMPLE 1.1 Kinematics of a planar pendulum

Consider the planar pendulum shown in Figure 1.3, where the orthogonal unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are fixed to the pendulum and they share the motion of the pendulum. The unit vectors $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$, which point along the X, Y and Z directions, respectively, are fixed in time. The velocity and acceleration of the planar pendulum P with respect to point O are the quantities of interest. The position vector from point O to point P is written as

$$\begin{aligned} \boldsymbol{r}^{P/O} &= \boldsymbol{r}^{Q/O} + \boldsymbol{r}^{P/Q} \\ &= h\boldsymbol{j} - L\boldsymbol{e}_2 \end{aligned} \tag{a}$$

Making use of Eqs. (1.2) and (1.8) and noting that both h and L are constant with respect to time and the angular velocity $\boldsymbol{\omega} = \dot{\theta}\boldsymbol{k}$, the pendulum velocity is

$$\begin{aligned} \boldsymbol{v}^{P/O} &= -L\frac{d\boldsymbol{e}_2}{dt} = -L\boldsymbol{\omega} \times \boldsymbol{e}_2 \\ &= -L\dot{\theta}\boldsymbol{k} \times \boldsymbol{e}_2 = L\dot{\theta}\boldsymbol{e}_1 \end{aligned} \tag{b}$$

and the pendulum acceleration is

$$\begin{aligned} \boldsymbol{a}^{P/O} &= \frac{d\boldsymbol{v}^{P/O}}{dt} \\ &= \frac{d(L\dot{\theta}\boldsymbol{e}_1)}{dt} = L\ddot{\theta}\boldsymbol{e}_1 + L\dot{\theta}\boldsymbol{\omega} \times \boldsymbol{e}_1 \\ &= L\ddot{\theta}\boldsymbol{e}_1 + L\dot{\theta}(\dot{\theta}\boldsymbol{k} \times \boldsymbol{e}_1) \\ &= L\ddot{\theta}\boldsymbol{e}_1 + L\dot{\theta}^2\boldsymbol{e}_2 \end{aligned} \tag{c}$$

In arriving at Eqs. (b) and (c), the following relations have been used.

$$\begin{aligned} \boldsymbol{k} \times \boldsymbol{e}_2 &= -\boldsymbol{e}_1 \\ \boldsymbol{k} \times \boldsymbol{e}_1 &= \boldsymbol{e}_2 \end{aligned} \tag{d}$$

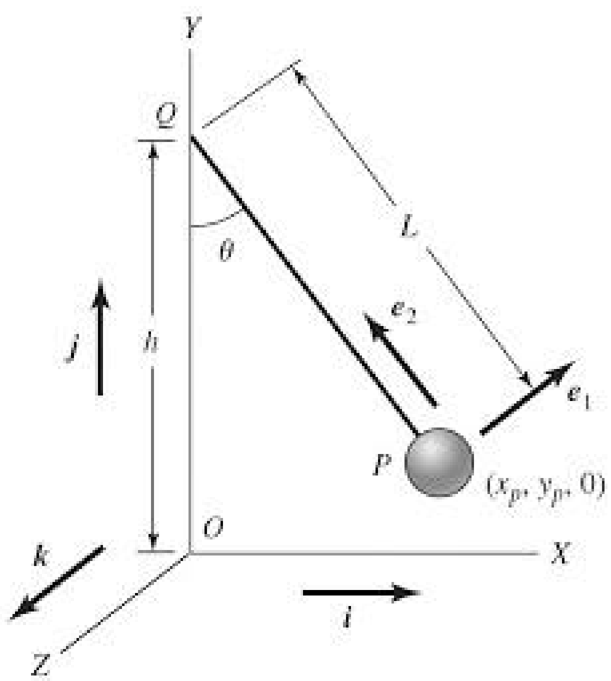

FIGURE 1.3
Planar pendulum. The reference frames are not explicitly shown in this figure, but it is assumed that the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are fixed in R and that the unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are fixed in R'.

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Since the orientation of the unit vectors $\boldsymbol{e}_1'$ and $\boldsymbol{e}_2'$ change with time due to the rotation of the disc, we have

$$\frac{d\boldsymbol{e}_1'}{dt} = \boldsymbol{\omega} \times \boldsymbol{e}_1' = \dot{\theta}\boldsymbol{k} \times \boldsymbol{e}_1' = \dot{\theta}\boldsymbol{e}_2'$$

$$\frac{d\boldsymbol{e}_2'}{dt} = \boldsymbol{\omega} \times \boldsymbol{e}_2' = \dot{\theta}\boldsymbol{k} \times \boldsymbol{e}_2' = -\dot{\theta}\boldsymbol{e}_1'$$

which leads to

$$\begin{aligned} \boldsymbol{V}_m &= -r\dot{\varphi}\sin\varphi\boldsymbol{e}_1' + (R + r\cos\varphi)\dot{\theta}\boldsymbol{e}_2' + r\dot{\varphi}\cos\varphi\boldsymbol{e}_2' - r\dot{\theta}\sin\varphi\boldsymbol{e}_1' \\ &= -r(\dot{\varphi} + \dot{\theta})\sin\varphi\boldsymbol{e}_1' + (R\dot{\theta} + r(\dot{\varphi} + \dot{\theta})\cos\varphi)\boldsymbol{e}_2' \end{aligned}$$

EXAMPLE 1.5 Moving mass on a rotating table[8]

Consider a mass m that is held by elastic constraints and is located on a table that is rotating at a constant speed ω, as shown in Figure 1.7. We shall determine the absolute velocity of the mass. We assume that point O is fixed in the vertical plane, that the unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are fixed to the mass m, as shown in Figure 1.7, and that $\boldsymbol{k} = \boldsymbol{e}_1 \times \boldsymbol{e}_2$. Then

$$\boldsymbol{r}_m = r\boldsymbol{e}_1$$

and the velocity is given by

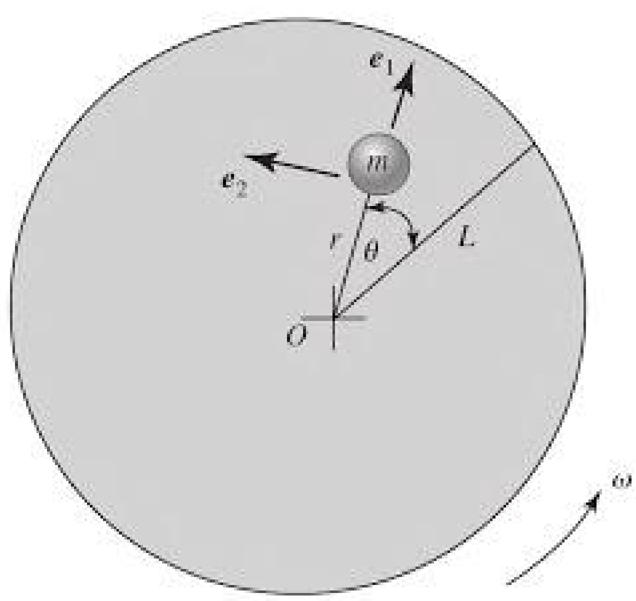

FIGURE 1.7
Frictionless rotating table of radius L on which mass m is elastically constrained.

[8] N. S. Clarke, "The Effect of Rotation upon the Natural Frequencies of a Mass-Spring System," *J. Sound Vibration*, **250**(5), pp. 849–887 (2000).

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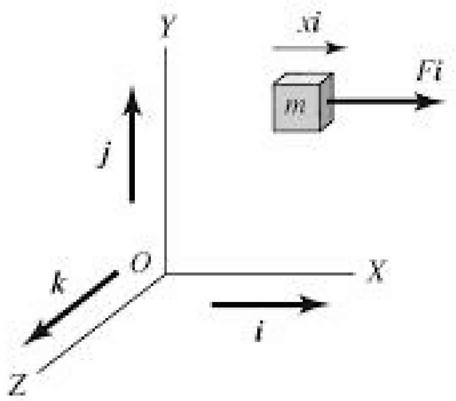

FIGURE 1.9
Free particle of mass m translating along the $\boldsymbol{i}$ direction.

$$\boldsymbol{F} = \frac{d(m\boldsymbol{v})}{dt} = m\boldsymbol{a} \tag{1.13}$$

which is referred to as *Newton's second law of motion*. The velocity in Eq. (1.12) and the acceleration in Eq. (1.13) are determined from kinematics. Therefore, for the particle shown in Figure 1.9, it follows from Eq. (1.13) that

$$F\boldsymbol{i} = m\ddot{x}\boldsymbol{i}$$

or

$$F = m\ddot{x}$$

Dynamics of a System of *n* Particles

For a system of n particles, the principle of linear momentum is written as

$$\begin{aligned} \boldsymbol{F} &= \sum_{i=1}^{n} \boldsymbol{F}_i = \sum_{i=1}^{n} \frac{d\boldsymbol{p}_i}{dt} = \frac{d\boldsymbol{p}}{dt} \\ &= \sum_{i=1}^{n} m_i \frac{d\boldsymbol{v}_i}{dt} \end{aligned} \tag{1.14}$$

where the subscript i refers to the ith particle in the collection of n particles, $\boldsymbol{F}_i$ is the external force acting on particle $\boldsymbol{i}$, $\boldsymbol{p}_i$ is the linear momentum of this particle, m_i is the constant mass of the ith particle, and $\boldsymbol{v}_i$ is the absolute velocity of the ith particle. For the jth particle in this collection, the governing equation takes the form

$$\boldsymbol{F}_j + \sum_{\substack{i=1 \\ i \neq j}}^{n} \boldsymbol{F}_{ij} = \frac{d\boldsymbol{p}_j}{dt} = m_j \frac{d\boldsymbol{v}_j}{dt} \tag{1.15}$$

where $\boldsymbol{F}_{ij}$ is the internal force acting on particle j due to particle i. Note that in going from the equation of motion for an individual particle given by Eq. (1.15) to that for a system of particles given by Eq. (1.14), it is assumed that all of the internal forces satisfy Newton's third law of motion; that is, the assumption of equal and opposite internal forces $(\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji})$.

If the center of mass of the system of particles is located at point G, then Eq. (1.14) can be shown to be equivalent to

$$\boldsymbol{F} = \frac{d(m\boldsymbol{v}_G)}{dt} \tag{1.16}$$

where m is the total mass of the system and $\boldsymbol{v}_G$ is the absolute velocity of the center of mass of the system. Equation (1.16) is also valid for a rigid body.

It is clear from Eq. (1.11) that *in the absence of external forces, the linear momentum of the system is conserved; that is, the linear momentum of the system is constant for all time*. This is an important conservation theorem, which is used, when applicable, to examine the results obtained from analysis of vibratory models.

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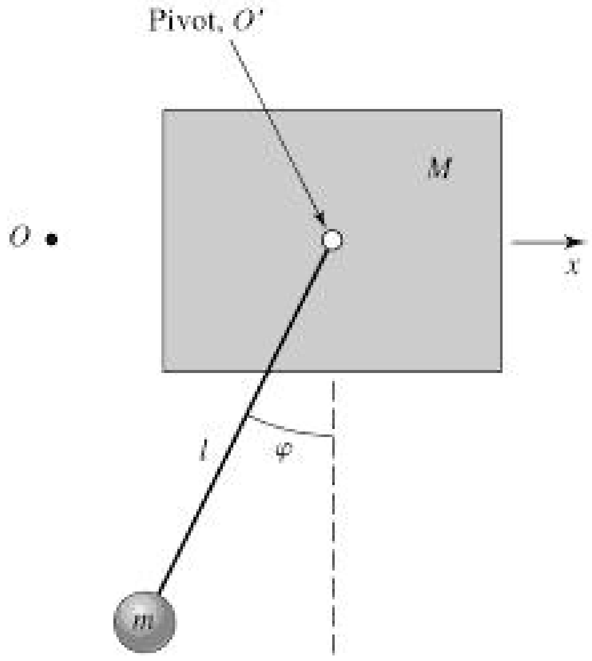
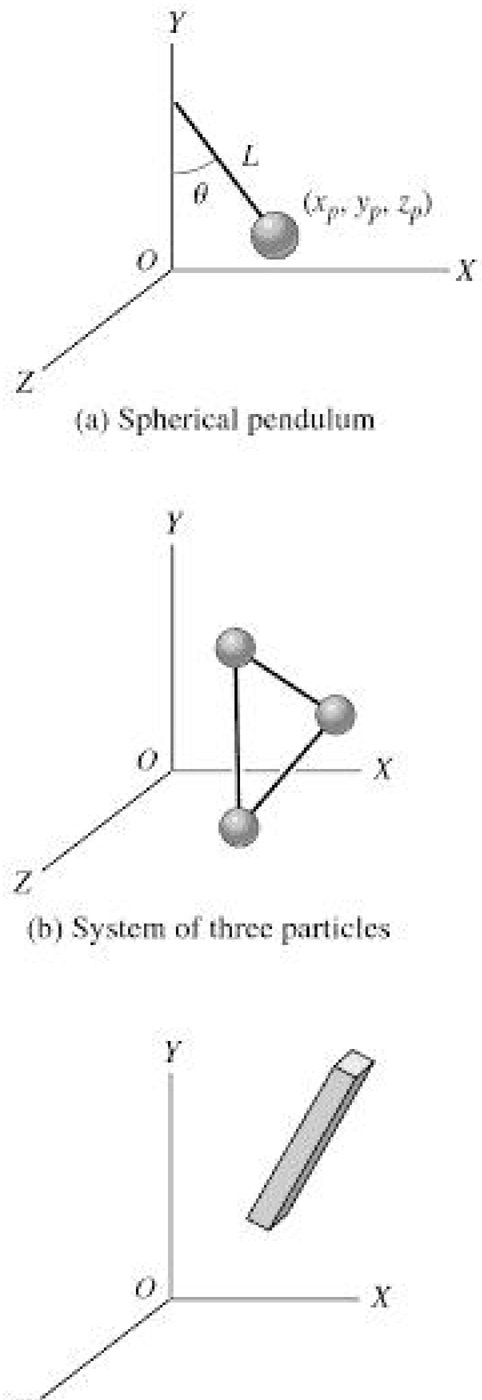


FIGURE E1.6

angle φ is used to describe the angular displacement of the pendulum from the vertical, determine the absolute velocity of the pendulum.

Section 1.2.2

1.7 Determine the number of degrees of freedom for the systems shown in Figure E1.7. Assume that the length L of the pendulum shown in Figure E1.7a is constant and that the length between each pair of particles in Figure E1.7b is constant. *Hint:* For Figure E1.7c, the rigid body can be thought of as a system of particles where the length between each pair of particles is constant.



(a) Spherical pendulum

(b) System of three particles

(c) Free rigid body in space

FIGURE E1.7

Section 1.2.3

1.8 Draw free-body diagrams for each of the masses shown in Figure E1.6 and obtain the equations of motion along the horizontal direction by using Eq. (1.15).

1.9 Draw the free-body diagram for the whole system shown in Figure E1.6, obtain the system equation of motion by using Eq. (1.14) along the horizontal direction, and verify that this equation can be obtained from Eq. (1.15).

1.10 Determine the linear momentum for the system shown in Figure E1.5 and discuss if it is conserved. Assume that the mass of the bar is M_{bar} and the distance from the point O to the center of the bar is L_{bar}.

1.11 Determine the angular momentum of the system shown in Figure 1.6 about the point O and discuss if it is conserved.

1.12 A rigid body is suspended from the ceiling by two elastic cables that are attached to the body at the points O' and O'', as shown in Figure E1.12. Point G is the center of mass of the body. Which of these points would you choose to carry out an angular-momentum balance based on Eq. (1.17)?

1.13 Consider the rigid body shown in Figure E1.13. This body has a mass m and rotary inertia J_G about the

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TABLE 2.1 Units of Components Comprising a Vibrating Mechanical System and Their Customary Symbols

Quantity	Units
Translational motion	
Mass, m	kg
Stiffness, k	N/m
Damping, c	N·s/m
External force, F	N
Rotational motion	
Mass moment of inertia, J	kg·m^2
Stiffness, k_t	N·m/rad
Damping, c_t	N·m·s/rad
External moment, M	N·m

The *inertia element* stores and releases kinetic energy, the *stiffness element* stores and releases potential energy, and the *dissipation* or *damping element* is used to express energy loss in a system. Each of these elements has different excitation-response characteristics and the excitation is in the form of either a force or a moment and the corresponding response of the element is in the form of a displacement, velocity, or acceleration. The inertia elements are characterized by a relationship between an applied force (or moment) and the corresponding acceleration response. The stiffness elements are characterized by a relationship between an applied force (or moment) and the corresponding displacement (or rotation) response. The dissipation elements are characterized by a relationship between an applied force (or moment) and the corresponding velocity response. The nature of these relationships, which can be linear or nonlinear, are presented in this chapter. The units associated with these elements and the commonly used symbols for the different elements are shown in Table 2.1.

In this chapter, we shall show how to:

- Compute the mass moment of inertia of rotational systems.
- Determine the stiffness of various linear and nonlinear elastic components in translation and torsion and the equivalent stiffness when many individual linear components are combined.
- Determine the stiffness of fluid, gas, and pendulum elements.
- Determine the potential energy of stiffness elements.
- Determine the damping for systems that have different sources of dissipation: viscosity, dry friction, fluid, and material.
- Construct models of vibratory systems.

2.2 INERTIA ELEMENTS

Translational motion of a mass is described as motion along the path followed by the center of mass. The associated inertia property depends only on the total mass of the system and is independent of the geometry of the mass distribution of the system. The inertia property of a mass undergoing rotational motions, however, is a function of the mass distribution, specifically the mass moment of inertia, which is usually defined about its center of mass or a fixed point O. When the mass oscillates about a fixed point O or a pivot point O, the rotary inertia J_O is given by

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and therefore, after making use of the parallel axis theorem, the mass moment of inertia about the point O is

$$J_O = J_G + m\left(\frac{L}{2}\right)^2 = \frac{1}{12}mL^2 + \frac{1}{4}mL^2 = \frac{1}{3}mL^2 \qquad \text{(b)}$$

EXAMPLE 2.2 Slider mechanism: system with varying inertia property

In Figure 2.3, a slider mechanism with a pivot at point O is shown. A slider of mass m_s slides along a uniform bar of mass m_l. Another bar, which is pivoted at point O', has a portion of length b that has a mass m_b and another portion of length e that has a mass m_e. We shall determine the rotary inertia J_O of this system and show its dependence on the angular displacement coordinate φ.

If a_e is the distance from the midpoint of bar of mass m_e to O and a_b is the distance from the midpoint of bar of mass m_b to O, then from geometry we find that

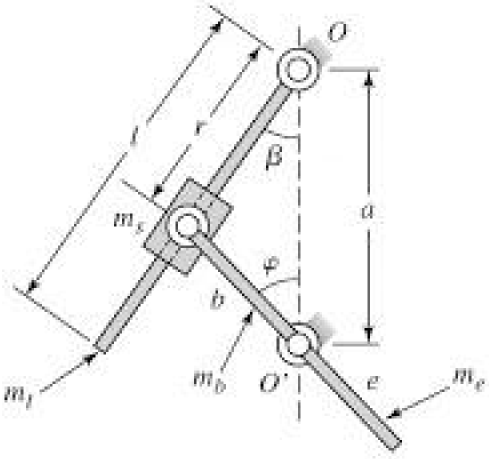

FIGURE 2.3
Slider mechanism.

$$\begin{aligned} r^2(\varphi) &= a^2 + b^2 - 2ab\cos\varphi \\ a_b^2(\varphi) &= (b/2)^2 + a^2 - ab\cos\varphi \\ a_e^2(\varphi) &= (e/2)^2 + a^2 - ae\cos(\pi - \varphi) \end{aligned} \qquad \text{(a)}$$

and hence, all motions of the system can be described in terms of the angular coordinate φ. The rotary inertia J_O of this system is given by

$$J_O = J_{m_l} + J_{m_s}(\varphi) + J_{m_b}(\varphi) + J_{m_e}(\varphi) \qquad \text{(b)}$$

where

$$\begin{aligned} J_{m_l} &= \frac{1}{3}m_l l^2, \qquad J_{m_s}(\varphi) = m_s r^2(\varphi) \\ J_{m_b}(\varphi) &= m_b\frac{b^2}{12} + m_b a_b^2 = m_b\left[\frac{b^2}{3} + a^2 - ab\cos\varphi\right] \\ J_{m_e}(\varphi) &= m_e\frac{e^2}{12} + m_e a_e^2 = m_e\left[\frac{e^2}{3} + a^2 - ae\cos(\pi - \varphi)\right] \end{aligned} \qquad \text{(c)}$$

In arriving at Eqs. (b) and (c), the parallel-axes theorem has been used in determining the bar inertias J_{m_b}, J_{m_e}, and J_{m_l}. From Eqs. (b) and (c), it is clear that the rotary inertia J_O of this system is a function of the angular displacement φ.

2.3 STIFFNESS ELEMENTS

2.3.1 Introduction

Stiffness elements are manufactured from different materials and they have many different shapes. One chooses the type of element depending on the requirements; for example, to minimize vibration transmission from machinery

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$$V(\theta) = \int_0^\theta \tau(\theta)d\theta = \int_0^\theta k_t\theta d\theta = \frac{1}{2}k_t\theta^2 \tag{2.12}$$

Combinations of Linear Springs

Different combinations of linear spring elements are now considered and the equivalent stiffness of these combinations is determined. First, combinations of translation springs shown in Figures 2.6b and 2.6c are considered and following that, combinations of torsion springs shown in Figures 2.7a and 2.7b are considered.

When there are two springs in parallel as shown in Figure 2.6b and the bar on which the force F acts remains parallel to its original position, then the displacements of both springs are equal and, therefore, the total force is

$$\begin{aligned} F(x) &= F_1(x) + F_2(x) \\ &= k_1x + k_2x = (k_1 + k_2)x = k_ex \end{aligned} \tag{2.13}$$

where $F_j(x)$ is the resulting force in spring k_j, $j = 1, 2$, and k_e is the equivalent spring constant for two springs in parallel given by

$$k_e = k_1 + k_2 \tag{2.14}$$

When there are two springs in series, as shown in Figure 2.6c, the force on each spring is the same and the total displacement is

$$\begin{aligned} x &= x_1 + x_2 \\ &= \frac{F}{k_1} + \frac{F}{k_2} = \left(\frac{1}{k_1} + \frac{1}{k_2}\right)F = \frac{F}{k_e} \end{aligned} \tag{2.15}$$

where the equivalent spring constant k_e is

$$k_e = \left(\frac{1}{k_1} + \frac{1}{k_2}\right)^{-1} = \frac{k_1k_2}{k_1 + k_2} \tag{2.16}$$

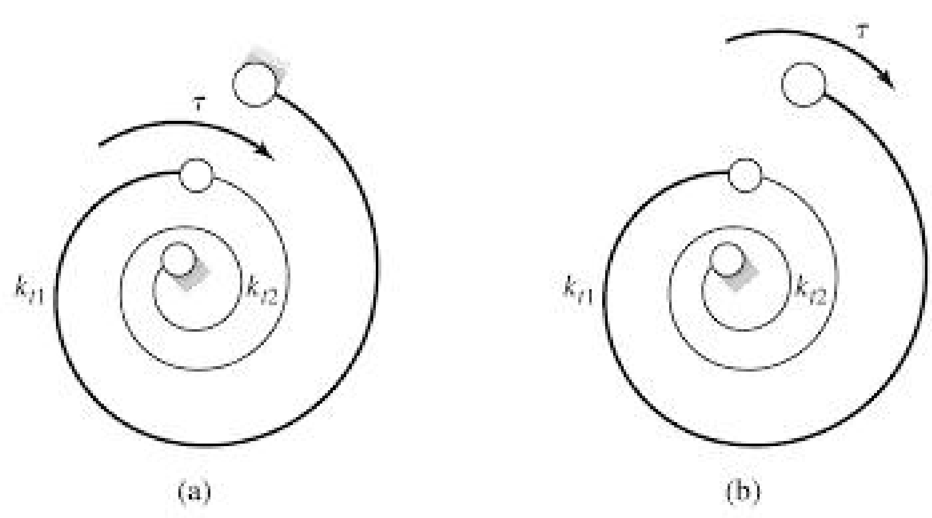

FIGURE 2.7
Two torsion springs: (a) parallel combination and (b) series combination.

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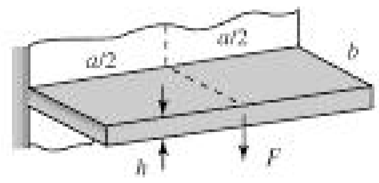
TABLE 2.3 (continued)

12	Cantilever plate, constant thickness, force at center of free edge[c]		$k = \dfrac{0.496Eh^3}{b^2(1-\nu^2)}$;	ν = Poisson's ratio, $a >> b$

A: area of cross section; *E*: Young's modulus; *G*: shear modulus; *I*: area moment of inertia or polar moment of inertia

[a]S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill, New York, (1959) p. 206.
[b]S. Timoshenko and S. Woinowsky-Krieger, *ibid*, p. 69.
[c]S. Timoshenko and S. Woinowsky-Krieger, *ibid*, p. 210.

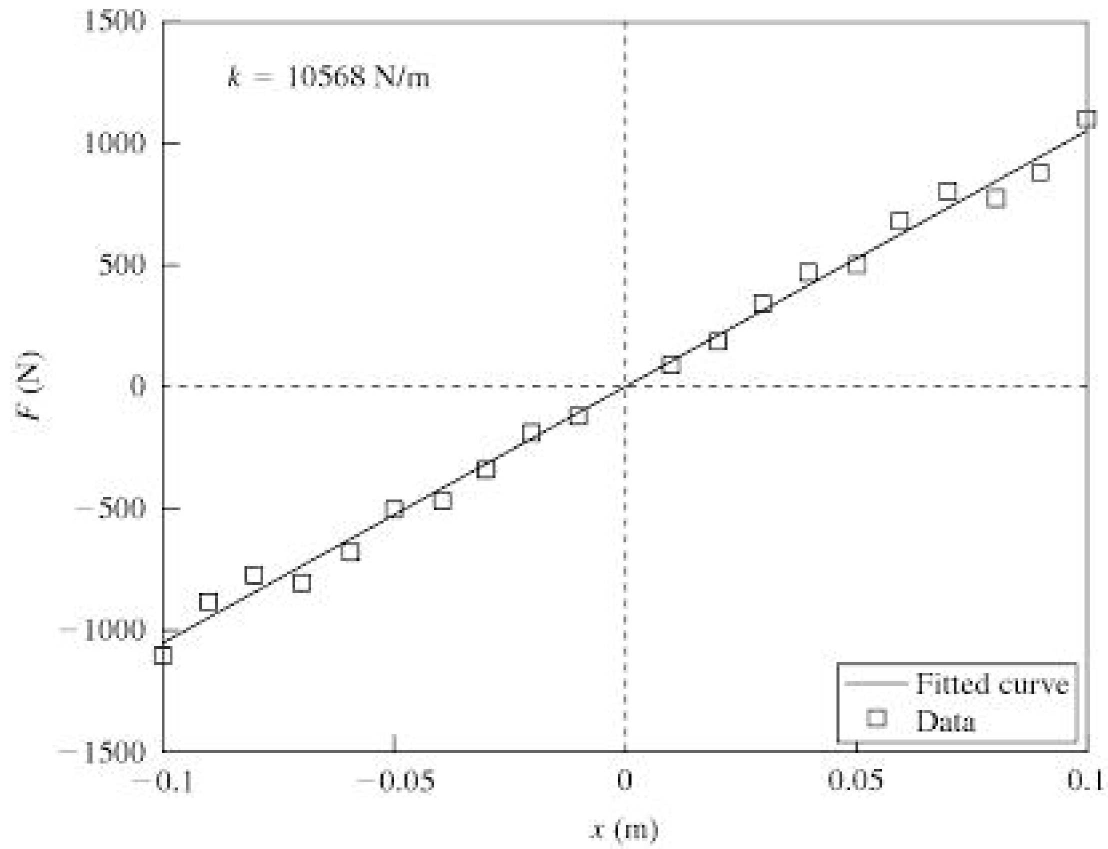


FIGURE 2.8
Experimentally obtained data used to determine the linear spring constant *k*.

parameter identification; identification and estimation of parameters of vibratory systems are addressed in the field of *experimental modal analysis*.[4] In experimental modal analysis, dynamic loading is used for parameter estimation. A further discussion is provided in Chapter 5, when system input-output relations (transfer functions and frequency response functions) are considered.

Next, some examples are considered to illustrate how the information shown in Table 2.3 can be used to determine equivalent spring constants for different physical configurations.

[4]D. J. Ewins, *Modal Testing: Theory and Practice*, John Wiley and Sons, NY (1984).

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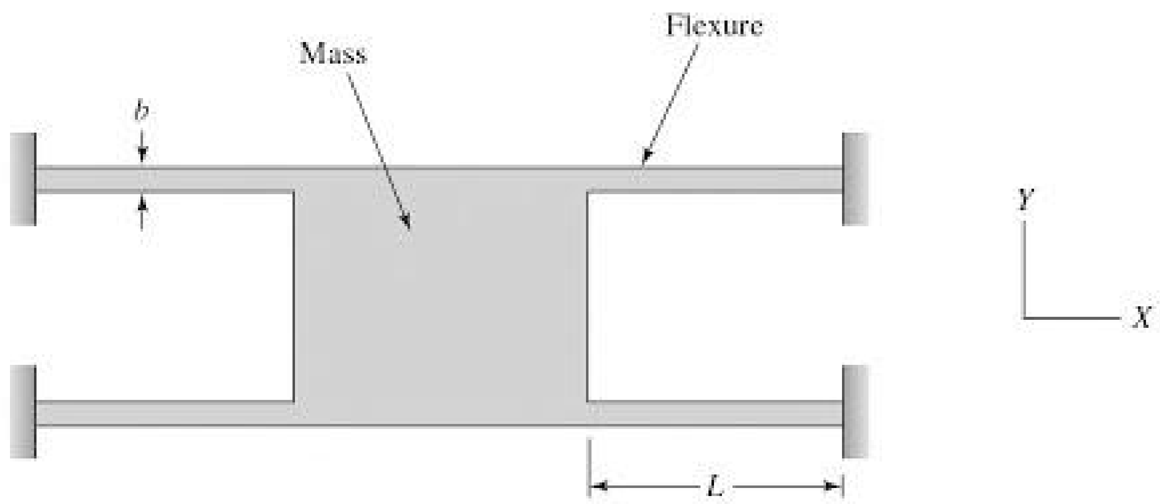

FIGURE 2.11
Fixed-fixed flexure used in a microelectromechanical system. *Source:* G.K.Fedder, "Simulation of Microelectromechanical Systems", Ph.D. Dissertation, Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA, (1994). Reprinted with permission of the author.

where the area moment of inertia is given by

$$I = \frac{bh^3}{12} \tag{b}$$

because the bending axis is along the Y direction. Since each of the four flexures experiences the same displacement at its end in the Z direction, this is a combination of four stiffness elements in parallel; hence, the equivalent stiffness of the system is given by

$$\begin{aligned} k_e &= 4 \times k_{\text{flexure}} \\ &= \frac{48\,EI}{L^3} \end{aligned} \tag{c}$$

Thus,

$$\begin{aligned} k_e &= \frac{48(150 \times 10^9)(2 \times 10^{-6})(2 \times 10^{-6})^3}{12(100 \times 10^{-6})^3}\ \text{N/m} \\ &= 9.6\ \text{N/m} \end{aligned} \tag{d}$$

EXAMPLE 2.7 Equivalent stiffness of springs in parallel: removal of a restriction

Let us reexamine the pair of springs in parallel shown in Figure 2.6b. Now, however, we remove the restriction that the bar to which the force is applied has to remain parallel to its original position. Then, we have the configuration shown in Figure 2.12. The equivalent spring constant for this configuration will be determined.

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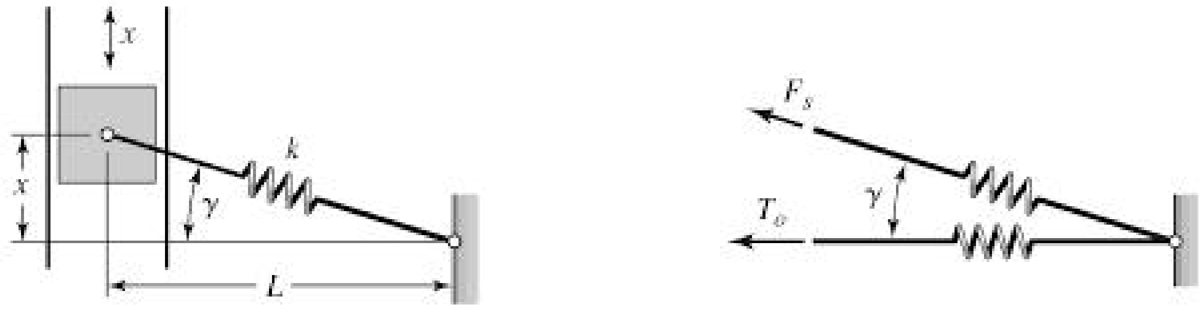

FIGURE 2.14
Nonlinear stiffness due to geometry: spring under an initial tension, one end of which is constrained to move in the vertical direction.

is initially under a tension force $T_o = k\delta_o$. When the spring is moved up or down an amount x in the vertical direction, the force in the spring is

$$F_s(x) = k\delta_o + k(\sqrt{L^2 + x^2} - L) \tag{a}$$

The force in the x-direction is obtained from Eq. (a) as

$$\begin{aligned} F_x(x) &= F_s \sin\gamma = \frac{F_s x}{\sqrt{L^2 + x^2}} \\ &= \frac{xk\delta_o}{\sqrt{L^2 + x^2}} + \frac{kx(\sqrt{L^2 + x^2} - L)}{\sqrt{L^2 + x^2}} \end{aligned} \tag{b}$$

which clearly shows that the spring force opposing the motion is a nonlinear function of the displacement x. Hence, a vibratory model of the system shown in Figure 2.14 will have nonlinear stiffness.

Cubic Springs and Linear Springs

If, in Eq. (b), we assume that $|x/L| << 1$ and expand the denominator of each term on the right-hand side of Eq. (b) as a binomial expansion and keep only the first two terms, we obtain

$$F_x(x) = k\delta_o \frac{x}{L} + \frac{k}{2}(L - \delta_o)\left(\frac{x}{L}\right)^3 \tag{c}$$

When the nonlinear term is negligible, Eq. (c) leads to the following linear relationship

$$F_x(x) = k\delta_o \frac{x}{L} = T_o \frac{x}{L} \tag{d}$$

From Eq. (d), it is seen that the spring constant is proportional to the initial tension in the spring.

Another example of a nonlinear spring is one that is piecewise linear as shown in Figure 2.15. Here, each spring is linear; however, as the deflection increases, another linear spring comes into play and the spring constant changes (increases). An illustration of the effects of this type of spring on a vibrating system is given in Section 4.5.1.

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or, from Eq. (2.36),

$$V(\theta) = \frac{mgL}{2}(1 - \cos\theta) \tag{2.37b}$$

When the angle of rotation θ about the upright position $\theta = 0$ is "small," we can use the Taylor series approximation[9]

$$\cos\theta = 1 - \frac{\theta^2}{2} + \cdots \tag{2.38}$$

and substitute this expression into Eq. (2.37b) to obtain

$$V(\theta) \approx \frac{1}{2}\left(\frac{mgL}{2}\right)\theta^2 = \frac{1}{2}k_e\theta^2 \tag{2.39}$$

where the equivalent spring constant is

$$k_e = \frac{mgL}{2} \tag{2.40}$$

Figure 2.18(b) In a similar manner, for "small" rotations about the upright position $\theta = 0$ in Figure 2.18b, we obtain the increase in potential energy for the system. Here it is assumed that a weightless bar supports the mass m_1. Choosing the reference position as the bottom position, we obtain

$$V(\theta) \approx \frac{1}{2}m_1gL\theta^2 = \frac{1}{2}k_e\theta^2 \tag{2.41}$$

where the equivalent spring constant is

$$k_e = m_1gL \tag{2.42}$$

In the configuration shown in Figure 2.18b, if the weightless bar is replaced by one that has a uniformly distributed mass m, then the total potential energy of the bar and the mass is

$$V(\theta) \approx \frac{1}{4}mgL\theta^2 + \frac{1}{2}m_1gL\theta^2 = \frac{1}{2}\left(\frac{m}{2} + m_1\right)gL\theta^2 = \frac{1}{2}k_e\theta^2 \tag{2.43}$$

where the equivalent spring constant is

$$k_e = \left(\frac{m}{2} + m_1\right)gL \tag{2.44}$$

Figure 2.18(c) When the pendulum is inverted as shown in Figure 2.18c, then there is a decrease in potential energy; that is,

$$V(\theta) \approx -\frac{1}{2}m_1gL\theta^2 \tag{2.45}$$

[9]T. B. Hildebrand, *Advanced Calculus for Applications*, Prentice Hall, Englewood Cliffs NJ (1976).

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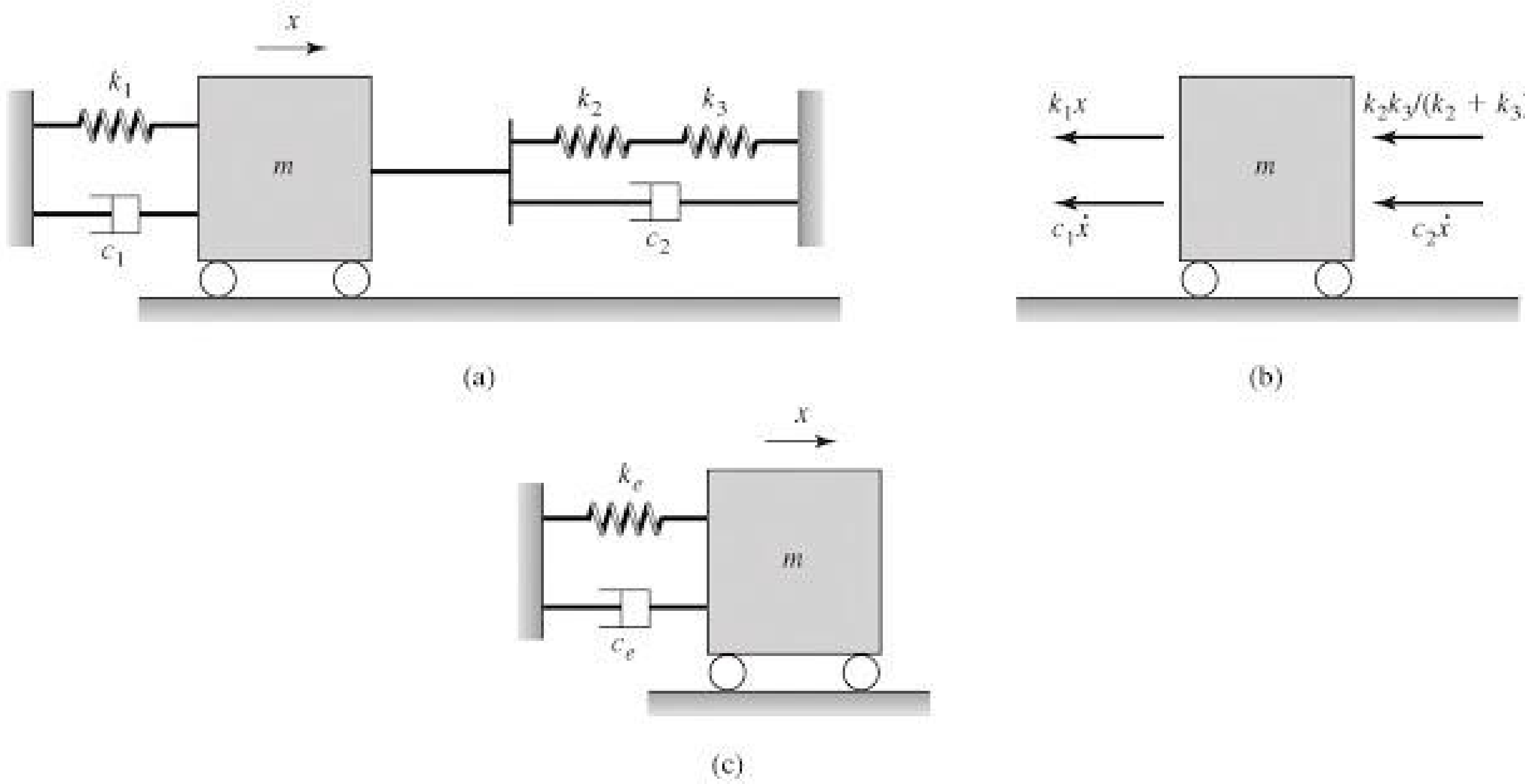

FIGURE 2.22
(a) Linear vibratory system; (b) free-body diagram of mass *m*; and (c) equivalent system.

In Figure 2.22c, the springs and dampers shown in Figure 2.22a have been collected and expressed as an equivalent spring and equivalent damper combination. Thus, we have

$$k_e = k_1 + \frac{k_2k_3}{k_2 + k_3}$$

$$c_e = c_1 + c_2$$

EXAMPLE 2.12 Equivalent linear damping coefficient of a nonlinear damper

It has been experimentally determined that the damper force-velocity relationship is given by the function

$$F(\dot{x}) = (4\ \text{N}\cdot\text{s/m})\dot{x} + (0.3\ \text{N}\cdot\text{s}^3/\text{m})\dot{x}^3 \quad \text{(a)}$$

We shall determine the equivalent linear damping coefficient around an operating speed of 3 m/s. To determine this damping coefficient, we use Eq. (2.47) and Eq. (a) and arrive at

$$\begin{aligned} c_e &= \left.\frac{dF(\dot{x})}{\dot{x}}\right|_{\dot{x}=3\ \text{m/s}} = 4\ \text{N}\cdot\text{s/m} + (0.9\ \text{N}\cdot\text{s}^3/\text{m}^3)\dot{x}^2\big|_{\dot{x}=3\ \text{m/s}} \\ &= 4\ \text{N}\cdot\text{s/m} + (0.9\ \text{N}\cdot\text{s}^3/\text{m}^3) \times (3^2\ \text{m}^2/\text{s}^2) \\ &= 12.1\ \text{N}\cdot\text{s/m} \end{aligned} \quad \text{(b)}$$

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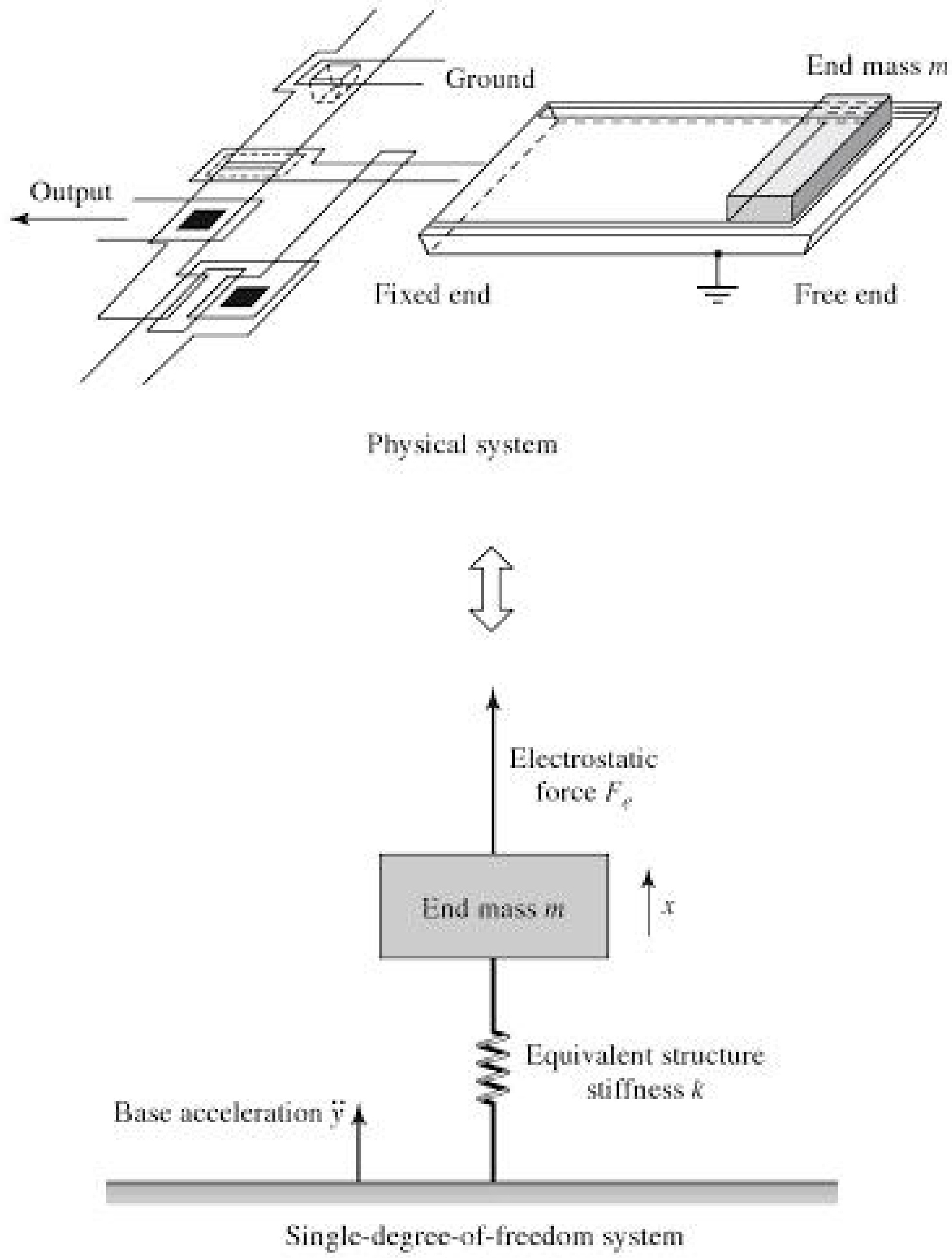
FIGURE 2.24
Microelectromechanical accelerometer and a vibratory model of this sensor. *Source:* From Systems Dynamics and Control 1st edition by Umez-Eronini. © 1999. Reprinted with permission of Nelson, a division of Thomson Learning: www.thomsonrights.com. Fax 800 730-2215.



2.5.3 The Human Body

In Figure 2.25, the human body and a vibratory model used to study the response of this physical system when subjected to vertical excitations are shown. While the vibratory model used in the previous section has only one discrete inertia element and one discrete spring element, the model[13] shown in Figure 2.25 has many inertial, spring, and damper elements.

Since many independent displacement variables are needed to describe the motion of this physical system, this vibratory model is an example of a system with multiple degrees of freedom. The response of systems with multiple degrees of freedom is treated in Chapters 7 and 8.

The human body is highly sensitive to vibration levels. While the body may sense displacements with amplitudes in the range of a hundredth of a mm, some of the components of the ear can sense even smaller displacements. In the low-frequency range from 1 Hz to 10 Hz, the perception of motion is said to be proportional to acceleration, and in the mid-frequency range

[13] M. P. Norton, *Fundamentals of Noise and Vibration Analysis for Engineers,* Cambridge University Press, New York (1989).

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fixed at both ends, while the beam model used for the work piece is considered fixed at one end and hinged or free at the other end where it is being cut.

In Section 4.4, the stability of a machine tool is determined from a vibratory model to avoid undesirable cutting conditions called chatter. This type of analysis can be used to choose parameters such as width of cut, spindle rpm, etc. In Chapter 9, vibrations of beams used in the model in Figure 2.27 are discussed at length.

2.6 DESIGN FOR VIBRATION

Principles that govern single degree-of-freedom systems, multiple degree-of-freedom systems, and continuous vibratory systems are covered in this book and are presented along with information needed to experimentally, numerically, and analytically investigate a vibratory system. In Figure 2.28, we show

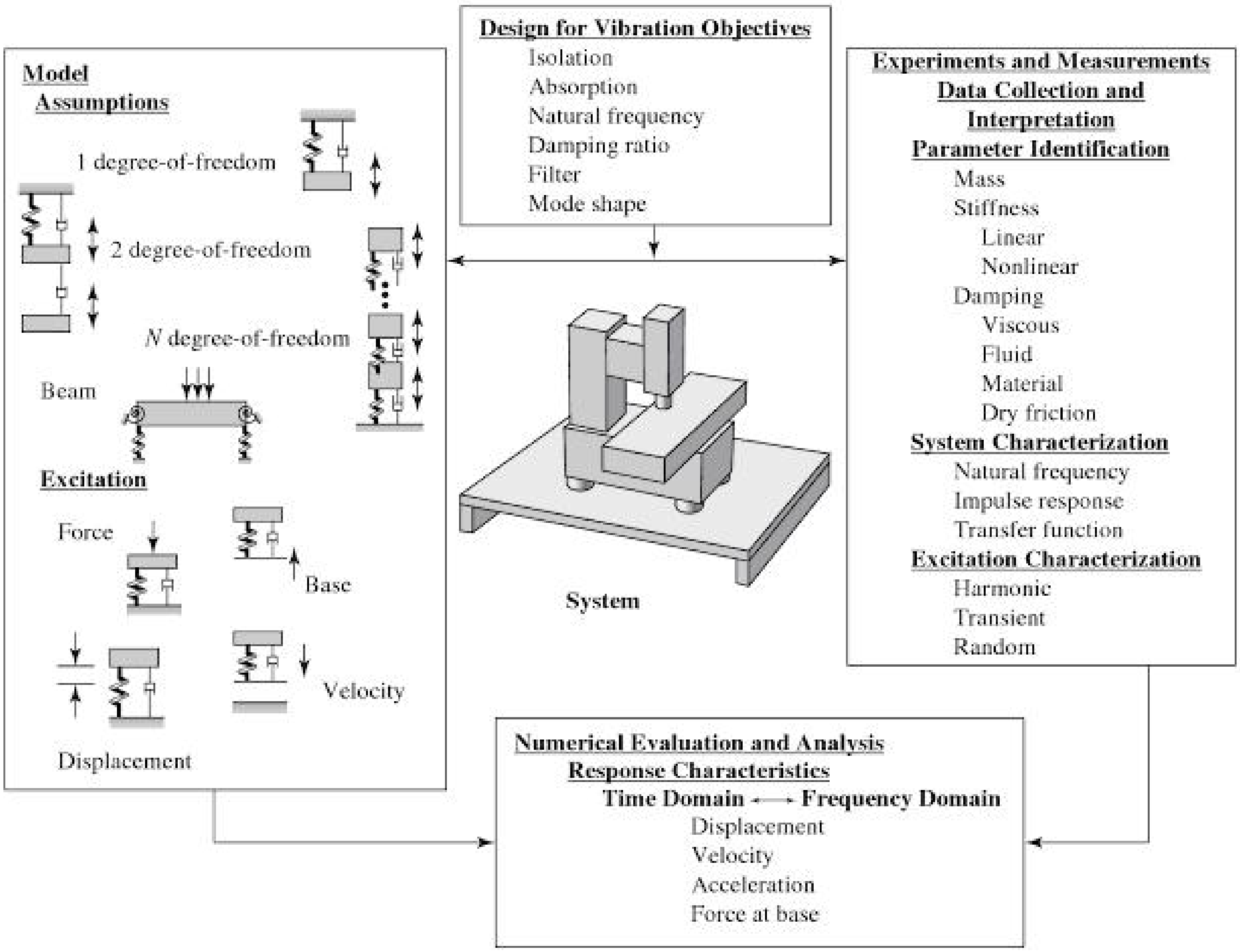

FIGURE 2.28
Design for vibration.

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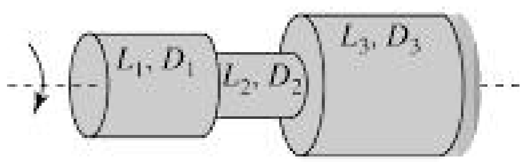

FIGURE E2.15

Section 2.3.3

2.16 Consider the two nonlinear springs in parallel that are shown in Figure E2.16. The force-displacement relations for each spring are, respectively,

$$F_j(x) = k_j x + k_j \alpha x^3 \qquad j = 1, 2$$

a) Obtain the expressions from which the equivalent spring constant can be determined.
b) If $F = 1000$ N, $k_1 = k_2 = 50{,}000$ N/m, and $\alpha = 2\ \text{m}^{-2}$, determine the equivalent spring constant.

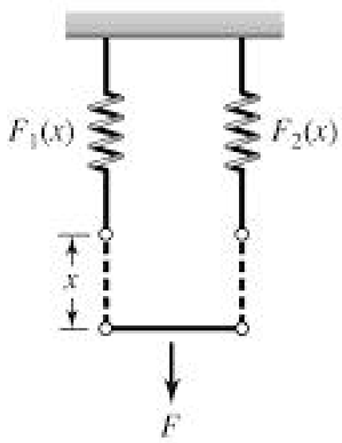

FIGURE E2.16

2.17 Consider the two nonlinear springs in series shown in Figure E2.17. The force-displacement relations for each spring are, respectively,

$$F_j(x) = k_j x + k_j \alpha x^3 \qquad j = 1, 2$$

a) Obtain the expressions from which the equivalent spring constant can be determined.
b) If $F = 1000$ N, $k_1 = 50{,}000$ N/m, $k_2 = 25{,}000$ N/m, and $\alpha = 2\ \text{m}^{-2}$, determine the equivalent spring constant.

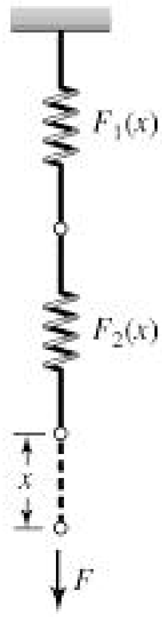

FIGURE E2.17

2.18 Consider the data in Table E2.18 in which the experimentally determined tire loads versus tire deflections have been recorded. These data are for a set of dual tires and a single wide-base tire.[18] The inflation pressure for all tires is 724 kN/m^2. Examine the stiffness characteristics of the two different tire systems and discuss them.

TABLE E2.18
Tire Load Versus Deflection Data

Tire Load (N)	Tire Deflection: Dual Tire (mm)	Tire Deflection: Single Wide-Base Tire (mm)
0	0	0
8896.4	7.62	10.2
17793	14	19
26689	19	27.9
35586	24.1	35.6
44482	27.9	41.9

Section 2.3.4

2.19 Consider the manometer shown in Figure 2.16 and seal the ends. Assume that the initial gas pressure of the sealed system is P_o and that L_o is the initial

[18] J. C. Tielking, "Conventional and wide base radial tyres," in Proceedings of the Third International Symposium on Heavy Vehicle Weights and Dimensions, D. Cebon and C. G. B. Mitchell, eds., Cambridge, UK, 28 June–2 July 1992, pp. 182–190.

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Springs, masses, and dampers are used to model vibrations systems. In the motorcycle, the coil spring in parallel with a viscous damper is attached to a mass composed of the tire and brake assembly. In the wind turbine, the mass of the propellers is supported by the column, which acts as the spring. (*Source:* Cohen/Ostrow / Getty Images; PKS Media, Inc. / Getty Images.)

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The directions of different forces along with their magnitudes are shown in Figure 3.1. Note that the inertia force $-m\ddot{x}\mathbf{j}$ is also shown along with the free-body diagram of the inertia element. Since the spring force is a restoring force and the damper force is a resistive force, they oppose the motion as shown in Figure 3.1. Based on Eq. (3.1b), we can carry out a force balance along the $\mathbf{j}$ direction and obtain the resulting equation

$$\underbrace{f(t)\mathbf{j} + mg\mathbf{j}}_{\substack{\text{External forces acting}\\ \text{on system}}} - \underbrace{(kx + k\delta_{st})\mathbf{j}}_{\substack{\text{Spring force acting}\\ \text{on mass}}} - \underbrace{c\frac{dr}{dt}\mathbf{j}}_{\substack{\text{Damping force}\\ \text{acting on mass}}} - \underbrace{m\frac{d^2r}{dt^2}\mathbf{j}}_{\text{Inertia force}} = 0 \tag{3.3}$$

Upon making use of Eq. (3.2), noting that L and δ_{st} are constants, and rearranging terms, Eq. (3.3) reduces to the following scalar differential equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + k(x + \delta_{st}) = f(t) + mg \tag{3.4}$$

Static-Equilibrium Position

The *static-equilibrium position* of a system is the position that corresponds to the system's rest state; that is, a position with zero velocity and zero acceleration. Dropping the time-dependent forcing term $f(t)$ and setting the velocity and acceleration terms in Eq. (3.4) to zero, we find that the static-equilibrium position is the solution of

$$k(x + \delta_{st}) = mg \tag{3.5}$$

If, in Eq. (3.5), we choose

$$\delta_{st} = \frac{mg}{k} \tag{3.6}$$

we find that $x = 0$ is the static-equilibrium position of the system. Equation (3.6) is interpreted as follows. Due to the weight of the mass m, the spring is stretched an amount δ_{st}, so that the spring force balances the weight mg. For this reason, δ_{st} is called the *static displacement*. Recalling that the spring has an unstretched length L, the static-equilibrium position measured from the origin O is given by

$$\mathbf{x}_{st} = x_{st}\mathbf{j} = (L + \delta_{st})\mathbf{j} \tag{3.7}$$

which is the rest position of the system. For the vibratory system of Figure 3.1, it is clear from Eq. (3.6) that the static-equilibrium position is determined by the spring force and gravity loading. An example of another type of static loading is provided in Example 3.1.

Equation of Motion for Oscillations about the Static-Equilibrium Position

Upon substituting Eq. (3.6) into Eq. (3.4), we obtain

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t) \tag{3.8}$$

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$$m\frac{d^2\hat{x}}{dt^2} - k\hat{x} = 0 \tag{h}$$

Comparing Eqs. (g) and (h), it is clear that the equations have different stiffness terms.

3.2.2 Moment-Balance Methods

For single degree-of-freedom systems that undergo rotational motion, such as the system shown in Figure 3.3, the moment balance method is useful in deriving the governing equation. A shaft with torsional stiffness k_t is attached to a disc with rotary inertia J_G about the axis of rotation, which is directed along the $\boldsymbol{k}$ direction. An external moment $M(t)$ acts on the disc, which is immersed in an oil-filled housing. Let the variable θ describe the rotation of the disc, and let the rotary inertia of the shaft be negligible in comparison to that of the disk.

The principle of angular momentum given by Eq. (1.17) is applied to obtain the equation governing the disc's motion. First the angular momentum $\boldsymbol{H}$ of the disc is determined. Since the disk is a rigid body undergoing rotation in the plane, Eq. (1.20) is used to write the angular momentum about the center of mass of the disc as

$$\boldsymbol{H} = J_G\dot{\theta}\boldsymbol{k}$$

Thus, since the rotary inertia J_G and the unit vector $\boldsymbol{k}$ do not change with time, Eq. (1.17) is rewritten as

$$\boldsymbol{M} - J_G\ddot{\theta}\boldsymbol{k} = 0 \tag{3.11}$$

where $\boldsymbol{M}$ is the total external moment acting on the free disk. Based on the free-body diagram shown in Figure 3.3, which also includes the inertial moment $-J_G\ddot{\theta}\boldsymbol{k}$, the governing equation of motion is

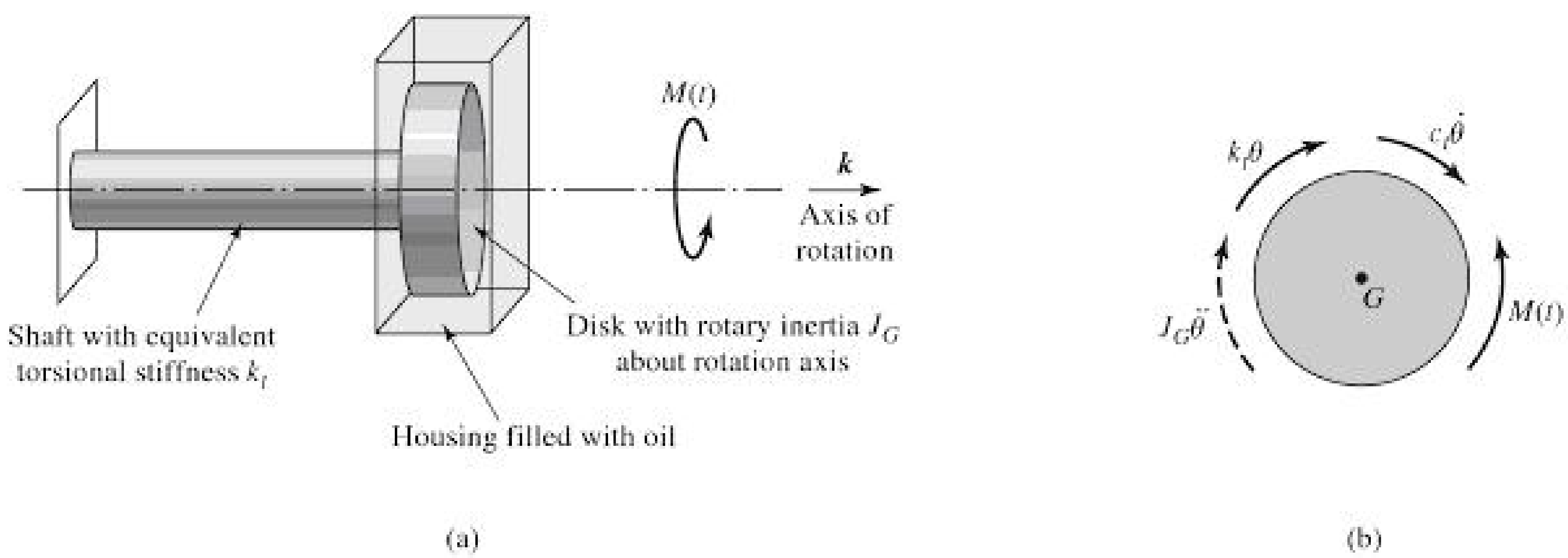

FIGURE 3.3
(a) A disc undergoing rotational motions and (b) free-body diagram of this disc in the plane normal to the axis of rotation.

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3.3.1 Natural Frequency

Translation Vibrations: Natural Frequency

For translation oscillations of a single degree-of-freedom system, the *natural frequency* ω_n of the system is defined as

$$\omega_n = 2\pi f_n = \sqrt{\frac{k}{m}} \text{ rad/s} \tag{3.14}$$

where k is the stiffness of the system and m is the system mass. The quantity f_n, which is also referred to as the natural frequency, has the units of Hz.

For the configuration shown in Figure 3.1, the vibratory system exhibits vertical oscillations. For such oscillations, we make use of Eq. (3.6) and Eq. (3.14) and obtain

$$\omega_n = 2\pi f_n = \sqrt{\frac{g}{\delta_{st}}} \text{ rad/s} \tag{3.15}$$

where δ_{st} is the static deflection of the system.

Rotational Vibrations: Natural Frequency

Drawing a parallel to the definition of natural frequency of translation motions of a single degree-of-freedom system, the natural frequency for rotational motions is defined as

$$\omega_n = 2\pi f_n = \sqrt{\frac{k_t}{J}} \text{ rad/s} \tag{3.16}$$

where k_t is the torsion stiffness of the system and J is the mass moment of inertia of the system.

Design Guideline: For single degree-of-freedom systems, an increase in the stiffness or a decrease in the mass or mass moment of inertia increases the natural frequency, whereas a decrease in the stiffness and/or an increase in the mass or mass moment of inertia decreases the natural frequency. Equivalently, when applicable, the greater the static displacement the lower the natural frequency; however, from practical considerations too large of a static displacement may be undesirable.

Period of Undamped Free Oscillations

For an unforced and undamped system, the *period of free oscillation* of the system is given by

$$T = \frac{1}{f_n} = \frac{2\pi}{\omega_n} \tag{3.17}$$

Thus, increasing the natural frequency decreases the period and vice versa.

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$$x_o = b\left(\frac{W}{a}\right)^{1/c} \tag{d}$$

For "small" amplitude vibrations about x_o, the linear equivalent stiffness of this spring is determined from Eqs. (c) and (d) to be

$$k_{eq} = \left.\frac{dF(x)}{dx}\right|_{x=x_o} = \frac{ac}{b}\left(\frac{x_o}{b}\right)^{c-1}$$
$$= \frac{ac}{b}\left(\frac{W}{a}\right)^{(c-1)/c} \tag{e}$$

Then, from Eqs. (3.14) and (e), we determine the natural frequency of this system as

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{W/g}} = \frac{1}{2\pi}\sqrt{\frac{gc}{b}\left(\frac{W}{a}\right)^{-1/c}}$$
$$= \frac{1}{2\pi}\sqrt{\frac{gc}{b}}\left(\frac{W}{a}\right)^{-1/(2c)} \text{Hz} \tag{f}$$

Representative Spring Data

We now consider the representative data of a nonlinear spring shown in Figure 3.5a. By using standard curve-fitting procedures,[7] we find that a = 2500 N, b = 0.011 m, and c = 2.77. After substituting these values into Eq. (f), we arrive at the natural frequency values shown in Figure 3.5b. It is seen that over a sizable portion of the load range, the natural frequency of the system varies within the range of ±8.8%. The natural frequency of a system with a linear spring whose static displacement ranges from 12 mm to 5 mm varies approximately from 4.5 Hz $(=\sqrt{9.8/0.012}/2\pi \text{ Hz})$ to 7.0 Hz $(=\sqrt{9.8/0.005}/2\pi \text{ Hz})$ or approximately ±22% about a frequency of 5.8 Hz.

3.3.2 Damping Factor

Translation Vibrations: Damping Factor

For translating single degree-of-freedom systems, such as those described by Eq. (3.8), the *damping factor* or *damping ratio* ζ is defined as

$$\zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}} = \frac{c\omega_n}{2k} \tag{3.18}$$

where c is the system damping coefficient with units of N·s/m, k is the system stiffness, and m is the system mass. The damping factor is a nondimensional quantity.

[7]The MATLAB function `lsqcurvefit` from the Optimization Toolbox was used.

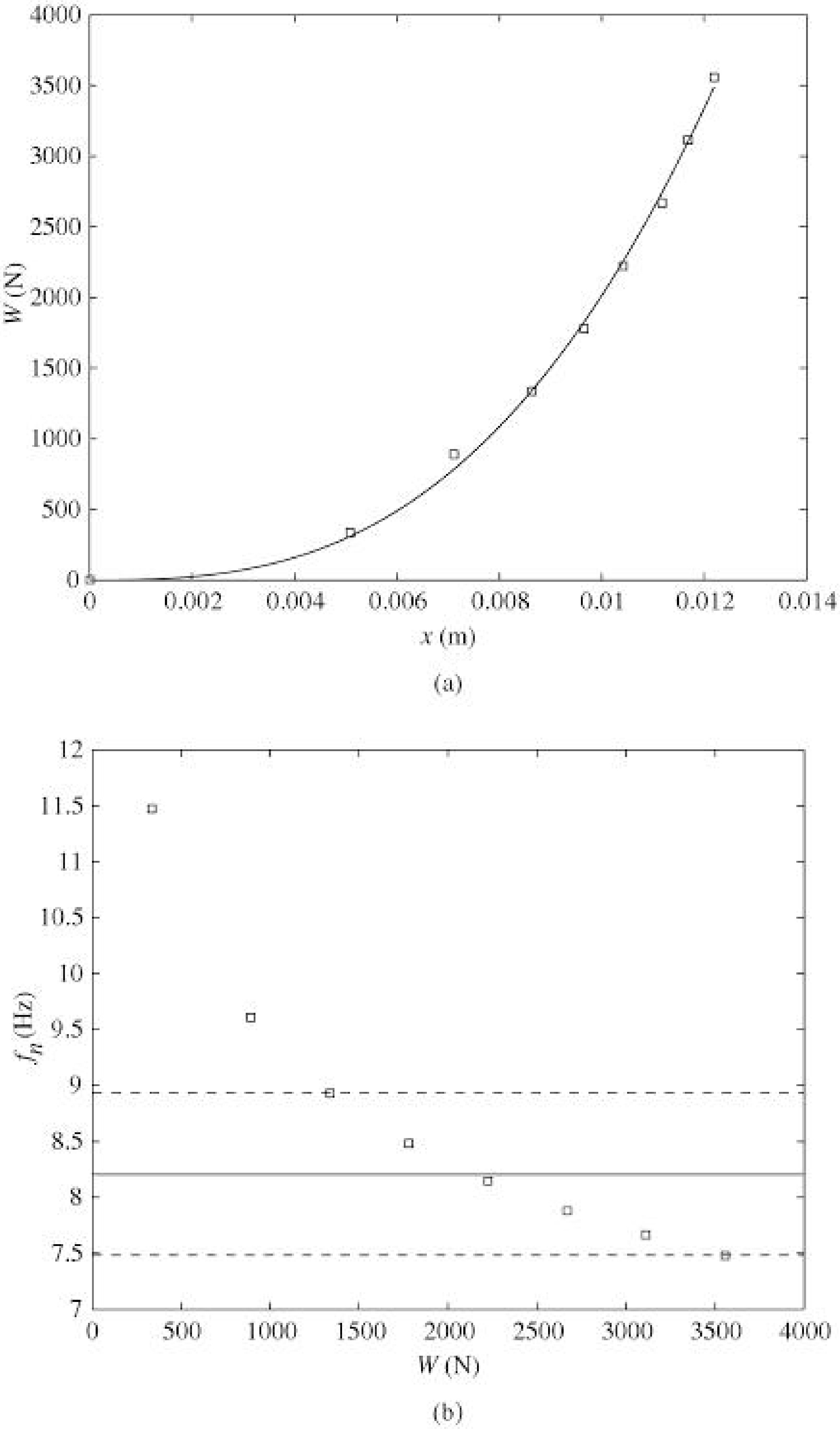

FIGURE 3.5

(a) Curve fit of nonlinear spring data: squares—experimental data values; solid line—fitted curve; (b) natural frequency for data values in (a) above: horizontal broken lines are within ±8.8% from the solid horizontal line.

Critical Damping, Underdamping, and Overdamping

Defining the quantity c_c, called the *critical damping*, as

$$c_c = 2m\omega_n = 2\sqrt{km} \tag{3.19}$$

the damping ratio is rewritten in the form

$$\zeta = \frac{c}{c_c} \tag{3.20}$$

When $c = c_c$, $\zeta = 1$. The significance of c_c is discussed in Section 4.2, where free oscillations of vibratory systems are considered. A system for which $0 < \zeta < 1$ is called an *underdamped* system and a system for which $\zeta > 1$ is called an *overdamped* system. A system for which $\zeta = 1$ is called a *critically* damped system.

Rotational Vibrations: Damping Factor

For rotating single degree-of-freedom systems such as those described by Eq. (3.13), the *damping ratio* ζ is defined as

$$\zeta = \frac{c_t}{2J\omega_n} = \frac{c_t}{2\sqrt{k_t J}} \tag{3.21}$$

where the damping coefficient c_t has the units N·m·s/rad.

From Eqs. (3.14) and (3.16), we see that the stiffness and inertia properties affect the natural frequency. From Eqs. (3.18) and (3.21), we see that the damping ratio is affected by any change in the stiffness, inertia, or damping property. However, one can change more than one system parameter in such a way that the net effect on ζ remains unchanged. This is shown in Example 3.8.

Governing Equation of Motion in Terms of Natural Frequency and Damping Factor

Introducing the definitions given by Eqs. (3.14) and (3.18) into Eq. (3.8), we obtain

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = \frac{f(t)}{m} \tag{3.22}$$

The significance of the quantities ω_n and ζ will become apparent when the solution to Eq. (3.22) is discussed in detail in the subsequent chapters. If we introduce the dimensionless time $\tau = \omega_n t$, then Eq. (3.22) can be written as

$$\frac{d^2x}{d\tau^2} + 2\zeta\frac{dx}{d\tau} + x = \frac{f(\tau)}{k} \tag{3.23}$$

It is seen from Eq. (3.23) that the natural frequency associated with the nondimensional system is always unity (one), and that the damping factor ζ is the only system parameter that appears explicitly on the left-hand side of the equation. We shall use both forms of Eqs. (3.22) and (3.23) in subsequent chapters.

In the absence of forcing, that is, when $f(\tau) = 0$, the motion of a vibratory system expressed in terms of nondimensional quantities can be described by just one system parameter. This fact is further elucidated in Section 4.2, where free oscillations are considered and it is shown that the qualitative nature of these oscillations can be completely characterized by the damping factor. In the presence of forcing, that is, $f(\tau) \neq 0$, both the damping factor ζ and the natural frequency ω_n are important for characterizing the nature of the response. This is further addressed when the forced responses of single degree-of-freedom systems are considered in Chapters 5 and 6.

Since the damping coefficient is one of the most important descriptors of a vibratory system, it is important to understand its interrelationships with the component's parameters m (or J), c (or c_t), and k (or k_t). We shall illustrate some of these relationships with the following example.

EXAMPLE 3.7 Effect of mass on the damping factor

A system is initially designed to be critically damped—that is, with a damping factor of $\zeta = 1$. Due to a design change, the mass of the system is increased 20%—that is, from m to $1.2m$. Will the system still be critically damped if the stiffness and the damping coefficient of the system are kept the same?

The definition of the damping factor is given by Eq. (3.18) and that for the critical damping factor is given by Eq. (3.19). Then, the damping factor of the system after the design change is given by

$$\zeta_{new} = \frac{c}{2\sqrt{k(1.2m)}} = 0.91\frac{c}{2\sqrt{km}} = 0.91\frac{c}{c_c} = 0.91 \times 1 = 0.91$$

Therefore, the system with the increased mass is no longer critically damped; rather, it is now underdamped.

EXAMPLE 3.8 Effects of system parameters on the damping ratio

An engineer finds that a single degree-of-freedom system with mass m, damping c, and spring constant k has too much static deflection δ_{st}. The engineer would like to decrease δ_{st} by a factor of 2, while keeping the damping ratio constant. We shall determine the different options.

Noting that this is a problem involving vertical vibrations, it is seen from Eqs. (3.6), (3.15), and (3.18) that

$$\delta_{st} = \frac{mg}{k}$$

$$2\zeta = \frac{c}{m}\sqrt{\frac{\delta_{st}}{g}} = c\sqrt{\frac{\delta_{st}}{gm^2}} = \frac{1}{m}\sqrt{\frac{c^2\delta_{st}}{g}} \quad \text{(a)}$$

From Eqs. (a), we see that there are three ways that one can achieve the goal.

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Furthermore, since

$$2\zeta = \frac{c}{\sqrt{km}} \tag{j}$$

the damping coefficient has to be reduced by a factor of $\sqrt{2}$; that is,

$$c \rightarrow \frac{c}{\sqrt{2}} \tag{k}$$

Thus, for $\delta_{st} \rightarrow \delta_{st}/2$ and k held constant, $m \rightarrow m/2$ and $c \rightarrow c/\sqrt{2}$.

Notice that in all three cases the natural frequency increases by a factor of $\sqrt{2}$. The results of this example can be generalized to a design guideline (see Exercises 3.22 and 3.23).

In the next two sections, the governing equations for different types of damping models and forcing conditions are presented. For all of these cases, translational motions are considered for illustrative purposes, and the equations are obtained by carrying out a force balance along the direction of motion. The form of the governing equations will be similar for systems involving rotational motions.

3.4 GOVERNING EQUATIONS FOR DIFFERENT TYPES OF DAMPING

The governing equations of motion for systems with different types of damping are obtained by replacing the term corresponding to the force due to viscous damping with the force due to either the fluid, structural, or dry friction type damping. Solutions for different periodically forced systems are given in Section 5.8, where equivalent viscous damping coefficients for different damping models are obtained.

Coulomb or Dry Friction Damping

After using Eq. (2.52) to replace the $c\dot{x}$ term in Eq. (3.8), the governing equation of motion takes the form

$$m\frac{d^2x}{dt^2} + kx + \underbrace{\mu mg\,\mathrm{sgn}(\dot{x})}_{\substack{\text{Nonlinear dry}\\ \text{friction force}}} = f(t) \tag{3.24}$$

which is a nonlinear equation because the damping characteristic is piece-wise linear. This piece-wise linear property can be used to find the solution of this system.

Fluid Damping

After using Eq. (2.54) to replace the $c\dot{x}$ term in Eq. (3.8), the governing equation takes the form

$$m\frac{d^2x}{dt^2} + \underbrace{c_d|\dot{x}|\dot{x}}_{\substack{\text{Nonlinear fluid} \\ \text{damping force}}} + kx = f(t) \tag{3.25}$$

which is a nonlinear equation due to the nature of the damping.

Structural Damping

After using Eq. (2.57) to replace the $c\dot{x}$ term in Eq. (3.8), we arrive at the governing equation

$$m\frac{d^2x}{dt^2} + k\beta\pi\text{sgn}(\dot{x})|x| + kx = f(t) \tag{3.26}$$

Equation (3.26) is further addressed in Section 5.8.

3.5 GOVERNING EQUATIONS FOR DIFFERENT TYPES OF APPLIED FORCES

In Section 3.2, we addressed governing equations of single degree-of-freedom systems whose inertial elements were subjected to direct excitations. Here, we address governing equations of single degree-of-freedom systems subjected to base excitations, systems excited by rotating unbalance, and systems immersed in a fluid.

3.5.1 System with Base Excitation

The base-excitation model is a prototype that is useful for studying buildings subjected to earthquakes, packaging during transportation, vehicle response, and for designing accelerometers (see Section 5.6). Here, the physical system of interest is represented by a single degree-of-freedom system whose base is subjected to a displacement disturbance $y(t)$, and an equation governing the motion of this system is sought to determine the response of the system $x(t)$.

If the system of interest is an automobile, then the road surface on which it is traveling can be a source of the disturbance $y(t)$ and the vehicle response $x(t)$ is to be determined. To avoid failure of electronic components during transportation, a base-excitation model is used to predict the vibration response of the electronic components. For buildings located above or adjacent to subways or above ground railroad tracks, the passage of trains can act as a source of excitation to the base of the building. In designing accelerometers, the accelerometer responses to different base excitations are studied to determine the appropriate accelerometer system parameters, such as the damping factor.

A prototype of a single degree-of-freedom system subjected to a base excitation is illustrated in Figure 3.6. The system represents an instrumentation package being transported in a vehicle. The vehicle provides the base excitation $y(t)$ to the instrumentation package modeled as a single degree-of-

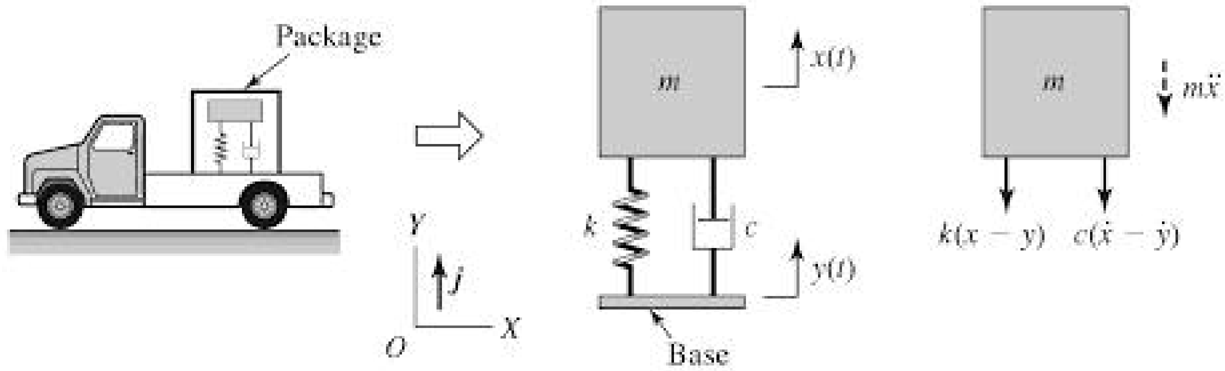

FIGURE 3.6
Base excitation and the free-body diagram of the mass.

freedom system. The displacement response $x(t)$ is measured from the system's static-equilibrium position. In the system shown in Figure 3.6, it is assumed that no external force is applied directly to the mass; that is, $f(t) = 0$. Based on the free-body diagram shown in Figure 3.6, we use Eq. (3.1b) to obtain the following governing equation of motion

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = c\frac{dy}{dt} + ky \tag{3.27}$$

which, on using Eqs. (3.14) and (3.18), takes the form

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y \tag{3.28}$$

The displacements $y(t)$ and $x(t)$ are measured from a fixed point O located in an inertial reference frame and a fixed point located at the system's static-equilibrium position, respectively. If the relative displacement is desired, then we let

$$z(t) = x(t) - y(t) \tag{3.29}$$

and Eq. (3.27) is written as

$$m\frac{d^2z}{dt^2} + c\frac{dz}{dt} + kz = -m\frac{d^2y}{dt^2} \tag{3.30}$$

while Eq. (3.28) becomes

$$\frac{d^2z}{dt^2} + 2\zeta\omega_n\frac{dz}{dt} + \omega_n^2 z = -\frac{d^2y}{dt^2} \tag{3.31}$$

where $\ddot{y}(t)$ is the acceleration of the base.

3.5.2 System with Unbalanced Rotating Mass

As discussed in Chapter 1, many rotating machines such as fans, clothes dryers, internal combustion engines, and electric motors, have a certain degree of unbalance. In modeling such systems as single degree-of-freedom systems, it is assumed that the unbalance generates a force that acts on the system's mass. This

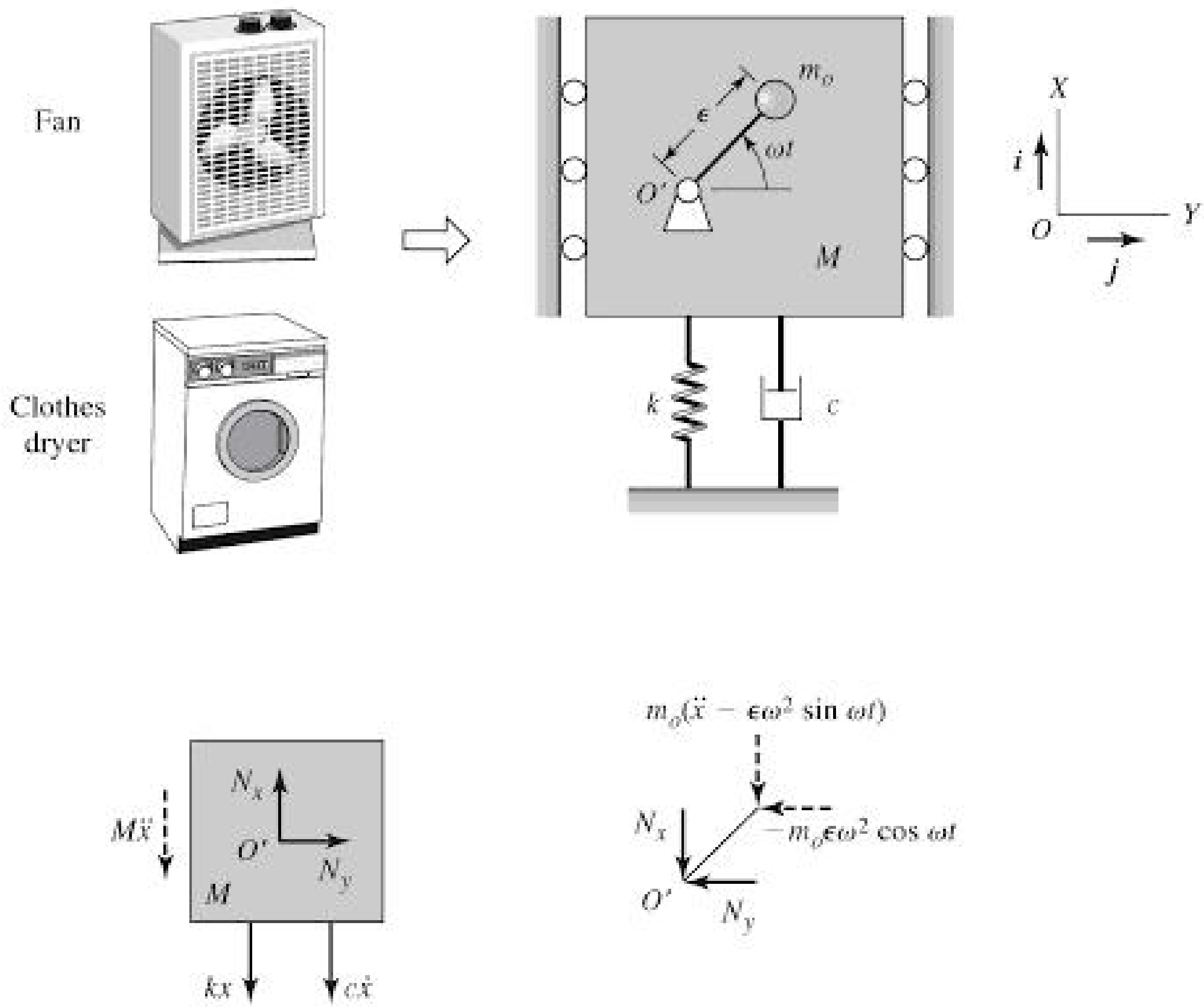

FIGURE 3.7
System with unbalanced rotating mass and free-body diagrams.

force, in turn, is transmitted through the spring and damper to the fixed base. The unbalance is modeled as a mass m_o that rotates with an angular speed ω, and this mass is located a fixed distance ϵ from the center of rotation as shown in Figure 3.7. Note that in Figure 3.7, M does not include the unbalance m_o.

For deriving the governing equation, only motions along the vertical direction are considered, since the presence of the lateral supports restrict motion in the $\boldsymbol{j}$ direction. The displacement of the system $x(t)$ is measured from the system's static-equilibrium position. The fixed point O is chosen to coincide with the vertical position of the static-equilibrium position. Based on the discussion in Section 3.2, gravity loading is not explicitly taken into account.

From the free-body diagram of the unbalanced mass m_o, we find that the reactions at the point O' are given by

$$\begin{aligned} N_x &= -m_o(\ddot{x} - \epsilon\omega^2 \sin \omega t) \\ N_y &= m_o \epsilon\omega^2 \cos \omega t \end{aligned} \tag{3.32}$$

and from the free-body diagram of mass M we find that

$$M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = N_x \tag{3.33}$$

Then, substituting for N_x from Eqs. (3.32) into Eq. (3.33), we arrive at the equation of motion

$$(M + m_o)\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = m_o\epsilon\omega^2 \sin \omega t \tag{3.34}$$

which is rewritten as

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = \frac{F(\omega)}{m}\sin \omega t \tag{3.35}$$

where

$$\begin{aligned} m &= M + m_o \\ \omega_n &= \sqrt{\frac{k}{m}} \\ F(\omega) &= m_o\,\epsilon\omega^2 \end{aligned} \tag{3.36}$$

In Eqs. (3.35) and (3.36), $F(\omega)$ is the magnitude of the unbalanced force. This magnitude depends on the unbalanced mass m_o and it is proportional to the square of the excitation frequency. From Eq. (3.6), it follows that the static displacement of the spring is

$$\delta_{st} = \frac{(M + m_o)g}{k} = \frac{mg}{k} \tag{3.37}$$

3.5.3 System with Added Mass Due to a Fluid

Consider a rigid body that is connected to a spring as shown in Figure 3.8. The entire system is immersed in a fluid. From Eq. (3.8) and Figure 3.8, and noting that $c = 0$ because there is no damper, the equation of motion of the system is

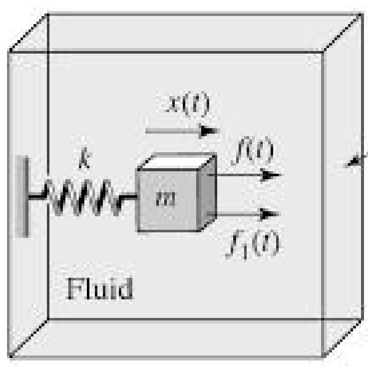

FIGURE 3.8
Vibrations of a system immersed in a fluid.

$$m\frac{d^2x}{dt^2} + kx = f(t) + f_1(t) \tag{3.38}$$

where $x(t)$ is measured from the unstretched position of the spring, $f(t)$ is the externally applied force, and $f_1(t)$ is the force exerted by the fluid on the mass due to the motion of the mass. The force generated by the fluid on the rigid body is[8]

$$f_1(t) = -K_o M\frac{d^2x}{dt^2} - C_f\frac{dx}{dt} \tag{3.39}$$

where M is the mass of the fluid displaced by the body, K_o is an added mass coefficient that is a function of the shape of the rigid body and the shape and size of the container holding the fluid, and C_f is a positive fluid damping coefficient that is a function of the shape of the rigid body, the kinematic viscosity of the fluid, and the frequency of oscillation of the rigid body.

[8] K. G. McConnell and D. F. Young, "Added mass of a sphere in a bounded viscous fluid," *J. Engrg. Mech. Div.*, Proc. ASCE, Vol. 91, No. 4, pp. 147–164 (1965).

Linear Single Degree-of-Freedom Systems

For linear single degree-of-freedom systems, the expressions for the system kinetic energy, the system potential energy, and the system dissipation function given by Eqs. (3.43) reduce to

$$T = \frac{1}{2}\sum_{j=1}^{1}\sum_{n=1}^{1} m_{jn}\dot{q}_j\dot{q}_n = \frac{1}{2}m_{11}\dot{q}_1^2$$
$$V = \frac{1}{2}\sum_{j=1}^{1}\sum_{n=1}^{1} k_{jn}q_jq_n = \frac{1}{2}k_{11}q_1^2$$
$$D = \frac{1}{2}\sum_{j=1}^{1}\sum_{n=1}^{1} c_{jn}\dot{q}_j\dot{q}_n = \frac{1}{2}c_{11}\dot{q}_1^2 \tag{3.46a}$$

Comparing the forms of the kinetic energy T, the potential energy V, and the dissipation function D with the standard forms given in Chapter 2, we find that

$$T = \frac{1}{2}m_e\dot{q}_1^2$$
$$V = \frac{1}{2}k_eq_1^2$$
$$D = \frac{1}{2}c_e\dot{q}_1^2 \tag{3.46b}$$

where m_e is the *equivalent mass*, k_e is the *equivalent stiffness*, and c_e is the *equivalent viscous damping;* they are given by

$$m_e = m_{11}$$
$$k_e = k_{11}$$
$$c_e = c_{11} \tag{3.46c}$$

On substituting Eqs. (3.46) into Eq. (3.44), the result is

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{q}_1}\left(\frac{1}{2}m_e\dot{q}_1^2\right)\right) - \frac{\partial}{\partial q_1}\left(\frac{1}{2}m_e\dot{q}_1^2\right) + \frac{\partial}{\partial\dot{q}_1}\left(\frac{1}{2}c_e\dot{q}_1^2\right) + \frac{\partial}{\partial q_1}\left(\frac{1}{2}k_eq_1^2\right) = Q_1$$
$$\frac{d}{dt}(m_e\dot{q}_1) - 0 + c_e\dot{q}_1 + k_eq_1 = Q_1$$
$$m_e\ddot{q}_1 + c_e\dot{q}_1 + k_eq_1 = Q_1 \tag{3.47}$$

Thus, to obtain the governing equation of motion of a linear vibrating system with viscous damping, one first obtains expressions for the system kinetic energy, system potential energy, and system dissipation function. If these quantities can be grouped so that an equivalent mass, equivalent stiffness, and equivalent damping can be identified, then, after the determination of the generalized force, the governing equation is given by the last of Eqs. (3.47). We see further from the definitions Eqs. (3.14) and (3.18) that

$$\omega_n = \sqrt{\frac{k_e}{m_e}}$$

$$\zeta = \frac{c_e}{2m_e\omega_n} = \frac{c_e}{2\sqrt{k_e m_e}} \tag{3.48}$$

It is noted that depending on the choice of the generalized coordinate, the determined equivalent inertia, equivalent stiffness, and equivalent damping properties of a system will be different. In the rest of this section, the use of the Lagrange equations is illustrated with eleven examples. As illustrated in these examples, we use the last of Eqs. (3.47) to obtain the governing equations of motion if the system kinetic energy, system potential energy, and dissipation function are in the form of Eqs. (3.46b); otherwise, we use Eq. (3.44) directly to obtain the governing equation of motion.

It is noted that only the system displacements and velocities are needed from the kinematics to use Lagrange's method whereas, to use the force-balance and moment-balance methods, one also needs system accelerations and has to deal with internal forces. In addition, with the increasing use of symbolic manipulation programs, it has become more common to have these programs derive the governing equations directly from the Lagrange's equations.

EXAMPLE 3.9 Equation of motion for a linear single degree-of-freedom system

For the linear system of Figure 3.1, the equation of motion is derived by using Lagrange's equations. After choosing the generalized coordinate to be x, we determine the system kinetic energy, the system potential energy, and the dissipation function for the system. From these quantities and the determined generalized force, the governing equation of motion of the system is established for motions about the static equilibrium position.

First, we identify the following

$$q_1 = x, \quad \boldsymbol{F}_l = f(t)\boldsymbol{j}, \quad \boldsymbol{r}_l = x\boldsymbol{j}, \quad \text{and} \quad \boldsymbol{M}_l = 0 \tag{a}$$

where $\boldsymbol{j}$ is the unit vector along the vertical direction. Making use of Eqs. (3.45) and (a), we determine the generalized force as

$$Q_1 = \sum_l \boldsymbol{F}_l \cdot \frac{\partial \boldsymbol{r}_l}{\partial q_j} + 0 = f(t)\boldsymbol{j} \cdot \frac{\partial x\boldsymbol{j}}{\partial x} = f(t) \tag{b}$$

From Eqs. (2.3) and (2.10), we find that the system kinetic energy and potential energy are, respectively,

$$T = \frac{1}{2}m\dot{x}^2$$

$$V = \frac{1}{2}kx^2 \tag{c}$$

and, from Eqs. (3.46), the dissipation function is

$$D = \frac{1}{2} c\dot{x}^2 \tag{d}$$

Comparing Eqs. (c) and (d) to Eqs. (3.46), we recognize that

$$m_e = m, \qquad k_e = k, \qquad \text{and} \qquad c_e = c \tag{e}$$

Hence, from Eqs. (e) and the last of Eqs. (3.47), the governing equation of motion has the form

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t) \tag{f}$$

which is identical to Eq. (3.8).

In the following examples, we show how the Lagrange equations can be used to derive the governing equations for a wide range of single degree-of-freedom systems.

EXAMPLE 3.10 Equation of motion for a system that translates and rotates

In Figure 3.9, a system that translates and rotates is illustrated. After choosing a generalized coordinate, we construct the system kinetic energy, the system potential energy, and the dissipation function, and then noting that they are in the form of Eqs. (3.46), we determine the equivalent inertia, equivalent stiffness, and equivalent damping coefficient. Based on these equivalent system properties and the last of Eqs. (3.47), we obtain the governing equation of motion of this system. We also determine the expressions for the natural frequency and the damping factor.

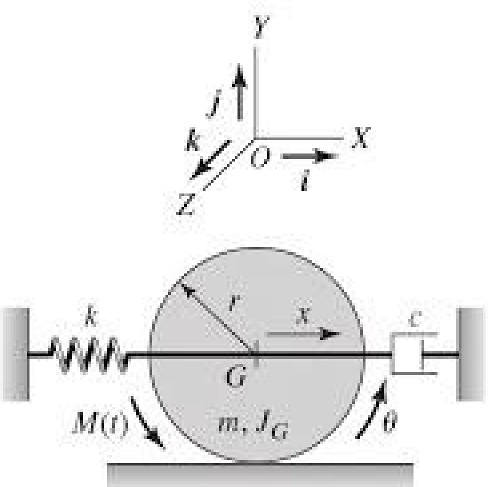

FIGURE 3.9
Disc rolling and translating.

As shown in Figure 3.9, the disc has a mass m and a mass moment of inertia J_G about its center G. The disc rolls without slipping. The horizontal location of the fixed point O is chosen to coincide with the unstretched length of the spring, and the horizontal translations of the center of mass of the disc are measured from this point O. When the center of the disc translates an amount x along the horizontal direction $\boldsymbol{i}$, then $x = r\theta$, where θ is the corresponding rotation of the disc about an axis parallel to $\boldsymbol{k}$. We can choose either x or θ as the generalized coordinate and express the one that is not chosen in terms of the other. Here, we choose θ as the generalized coordinate. Furthermore, we recognize that

$$q_1 = \theta, \quad \boldsymbol{F}_l = 0, \quad \boldsymbol{M}_l = M(t)\boldsymbol{k}, \quad \text{and} \quad \boldsymbol{\omega}_l = \dot{\theta}\boldsymbol{k} \tag{a}$$

Then making use of Eqs. (3.45) and (a), we determine the generalized force to be

$$Q_1 = \sum_l \boldsymbol{M}_l \cdot \frac{\partial \boldsymbol{\omega}_l}{\partial \dot{q}_1} = M(t)\boldsymbol{k} \cdot \frac{\partial \dot{\theta}}{\partial \dot{\theta}}\boldsymbol{k} = M(t) \tag{b}$$

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EXAMPLE 3.11 Governing equation for an inverted pendulum

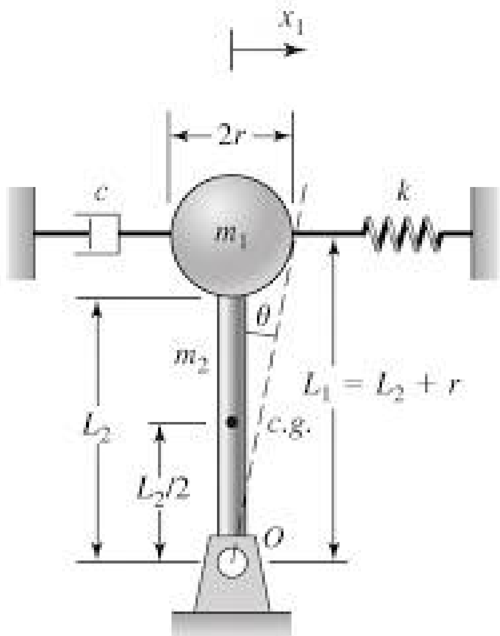

FIGURE 3.10
Inverted planar pendulum restrained by a spring and a viscous damper.

For the inverted pendulum shown in Figure 3.10, we obtain the governing equation of motion for "small" oscillations about the upright position. The natural frequency of the inverted pendulum is also determined and the natural frequency of a related pendulum system is examined. In the system of Figure 3.10, the bar, which carries the sphere of mass m_1, has a mass m_2 that is uniformly distributed along its length. A linear spring of stiffness k and a linear viscous damper with a damping coefficient c are attached to the sphere.

Before constructing the system kinetic energy, we determine the mass moments of inertia of the sphere of mass m_1 and the bar of mass m_2 about the point O. The total rotary inertia of the system is given by

$$J_O = J_{O1} + J_{O2} \tag{a}$$

where J_{O1} is the mass moment of inertia of m_1 about point O and J_{O2} is the mass moment of inertia of the bar about point O. Making use of Table 2.2 and the parallel-axes theorem, we find that

$$\begin{aligned} J_{O1} &= \frac{2}{5}m_1r^2 + m_1L_1^2 \\ J_{O2} &= \frac{1}{12}m_2L_2^2 + m_2\left(\frac{L_2}{2}\right)^2 = \frac{1}{3}m_2L_2^2 \end{aligned} \tag{b}$$

After choosing $q_1 = \theta$ as the generalized coordinate, and making use of Eqs. (a) and (b), we find that the system kinetic energy takes the form

$$\begin{aligned} T &= \frac{1}{2}J_O\dot{\theta}^2 = \frac{1}{2}[J_{O1} + J_{O2}]\dot{\theta}^2 \\ &= \frac{1}{2}\left[\frac{2}{5}m_1r^2 + m_1L_1^2 + \frac{1}{3}m_2L_2^2\right]\dot{\theta}^2 \end{aligned} \tag{c}$$

For "small" rotations about the upright position, we can express the translation of mass m_1 as

$$x_1 \approx L_1\theta \tag{d}$$

Then, making use of Eqs. (2.10), (2.39), (2.45), and (d), the system potential energy is constructed as

$$\begin{aligned} V &= \frac{1}{2}kx_1^2 - \frac{1}{2}m_1gL_1\theta^2 - \frac{1}{2}m_2g\frac{L_2}{2}\theta^2 \\ &= \frac{1}{2}\left[kL_1^2 - m_1gL_1 - m_2g\frac{L_2}{2}\right]\theta^2 \end{aligned} \tag{e}$$

The dissipation function takes the form

$$D = \frac{1}{2}c\dot{x}_1^2 = \frac{1}{2}cL_1^2\dot{\theta}^2 \tag{f}$$

Comparing Eqs. (c), (e), and (f) to Eqs. (3.46), we find that the equivalent inertia, the equivalent stiffness, and the equivalent damping properties of the system are given by, respectively,

$$
\begin{aligned}
m_e &= \frac{2}{5} m_1 r^2 + m_1 L_1^2 + \frac{1}{3} m_2 L_2^2 \\
k_e &= kL_1^2 - m_1 g L_1 - m_2 g \frac{L_2}{2} \\
c_e &= cL_1^2
\end{aligned} \tag{g}
$$

Noting that the only external force acting on the system is gravity loading, and that this has already been taken into account, the governing equation of motion is obtained from the last of Eqs. (3.47) as

$$m_e \ddot{\theta} + c_e \dot{\theta} + k_e \theta = 0 \tag{h}$$

Then, from the first of Eqs. (3.48) and (g), we find that

$$\omega_n = \sqrt{\frac{k_e}{m_e}} = \sqrt{\frac{kL_1^2 - m_1 g L_1 - m_2 g \dfrac{L_2}{2}}{J_{O1} + J_{O2}}} \tag{i}$$

It is pointed out that k_e can be negative, which affects the stability of the system as discussed in Section 4.3. The equivalent stiffness k_e is positive when

$$kL_1^2 > m_1 g L_1 + m_2 g \frac{L_2}{2} \tag{j}$$

that is, when the net moment created by the gravity loading is less than the restoring moment of the spring.

Natural Frequency of Pendulum System

In this case, we locate the pivot point O in Figure 3.10 on the top, so that the sphere is now at the bottom. The spring combination is still attached to the sphere. Then this pendulum system resembles the combination of the systems shown in Figures 2.17a and 2.17b. The equivalent stiffness of this system takes the form

$$k_e = kL_1^2 + m_1 g L_1 + m_2 g \frac{L_2}{2} \tag{k}$$

Noting that the equivalent inertia of the system is the same as in the inverted-pendulum case, we find the natural frequency of this system is

$$\omega_n = \sqrt{\frac{k_e}{m_e}} = \sqrt{\frac{kL_1^2 + m_1 g L_1 + m_2 g \dfrac{L_2}{2}}{J_{O1} + J_{O2}}} \tag{l}$$

If $m_2 << m_1$, $r << L_1$, and $k = 0$, then from Eqs. (b) and (l), we arrive at

$$\omega_n = \sqrt{\frac{m_1 g L_1 \left(1 + \dfrac{m_2 L_2}{m_1 L_1}\right)}{m_1 L_1^2 \left(1 + \dfrac{2r^2}{5L_1^2}\right)}} \rightarrow \sqrt{\frac{g}{L_1}} \tag{m}$$

which is the natural frequency of a pendulum composed of a rigid, weightless rod carrying a mass a distance L_1 from its pivot. We see that the natural frequency is independent of the mass and inversely proportional to the length L_1.

EXAMPLE 3.12 Governing equation for motion of a disk segment

For the half-disk shown in Figure 3.11, we will choose the coordinate θ as the generalized coordinate and establish the governing equation for the disc. Through this example, we illustrate how the system kinetic energy and the system potential energy can be approximated for "small" amplitude angular oscillations, so that the final form of the governing equation is linear. During the course of obtaining the governing equation, we determine the equivalent mass and equivalent stiffness of this system. The natural frequency of the disc is determined and it is shown that disc can be treated as a pendulum with a certain effective length. After determining the equivalent system properties, we determine the governing equation of motion based on the last of Eqs. (3.47).

As shown in Figure 3.11, the half-disk has a mass m and a mass moment of inertia J_G about the center of mass G. The system is assumed to oscillate without slipping. The point O is a fixed point, and the point of contact C is a distance $R\theta$ from the fixed point for an angular motion θ. The orthogonal unit vectors $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ are fixed in an inertial reference frame. The position vector from the fixed point O to the center of mass G is given by

$$\boldsymbol{r} = (-R\theta + b\sin\theta)\boldsymbol{i} + (R - b\cos\theta)\boldsymbol{j} \tag{a}$$

and the absolute velocity of the center of mass is determined from Eq. (a) to be

$$\dot{\boldsymbol{r}} = -(R - b\cos\theta)\dot{\theta}\boldsymbol{i} + b\sin\theta\dot{\theta}\boldsymbol{j} \tag{b}$$

Then, using Eq. (1.24) and selecting the generalized coordinate $q_1 = \theta$, the system kinetic energy takes the form

$$\begin{aligned} T &= \frac{1}{2}J_G\dot{\theta}^2 + \frac{1}{2}m(\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}) \\ &= \frac{1}{2}J_G\dot{\theta}^2 + \frac{1}{2}m[(R - b\cos\theta)^2 + b^2\sin^2\theta]\dot{\theta}^2 \\ &= \frac{1}{2}J_G\dot{\theta}^2 + \frac{1}{2}m[R^2 + b^2 - 2bR\cos\theta]\dot{\theta}^2 \end{aligned} \tag{c}$$

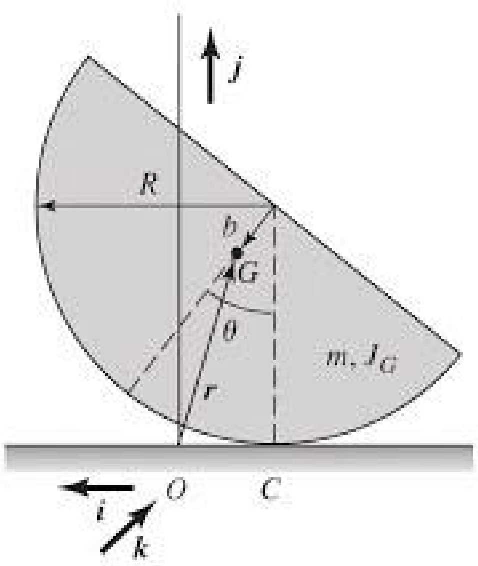

FIGURE 3.11
Half-disk rocking on a surface.

Choosing the fixed ground as the datum, the system potential energy takes the form

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Since the gravity loading has already been taken into account, the generalized force is zero. Furthermore, since there is no damping, the equivalent damping coefficient c_e is zero. Hence, from Eqs. (3.47) and (i), we arrive at the governing equation

$$[J_G + m(R-b)^2]\ddot{\hat{\theta}} + mgb\hat{\theta} = 0 \tag{j}$$

Natural Frequency

From Eq. (j), we find that the natural frequency is

$$\omega_n = \sqrt{\frac{mgb}{J_G + m(R-b)^2}} = \sqrt{\frac{g}{[J_G + m(R-b)^2]/mb}} \tag{k}$$

On comparing the form of Eq. (k) to the form of the equation for the natural frequency of a planar pendulum of length L_1 given by Eq. (m) of Example 3.11, we note that Eq. (k) is similar in form to the natural frequency of a pendulum with an effective length

$$L_e = \frac{J_G + m(R-b)^2}{mb} \tag{l}$$

EXAMPLE 3.13 Governing equation for a translating system with a pretensioned or precompressed spring

We revisit Example 2.8, and use Lagrange's equations to derive the governing equation of motion for vertical translations x of the mass about the static-equilibrium position of the system. Through this process, we will examine how the horizontal spring with linear stiffness k_1 affects the vibrations. The natural frequency of this system is also determined. The equation of motion will be derived for "small" amplitude vertical oscillations; that is, $|x/L| << 1$.

In the initial position, the horizontal spring is pretensioned with a tension T_1 as shown in Figure 3.12, which is produced by an initial extension of the spring by an amount δ_o; that is,

$$T_1 = k_1\delta_o \tag{a}$$

The kinetic energy of the system is

$$T = \frac{1}{2}m\dot{x}^2 \tag{b}$$

Next, we note that the potential energy is given by

$$V = V_1 + V_2 \tag{c}$$

where V_i, $i = 1, 2$, is the potential energy associated with the spring of stiffness k_i. Note that gravitational loading is not taken into account because we

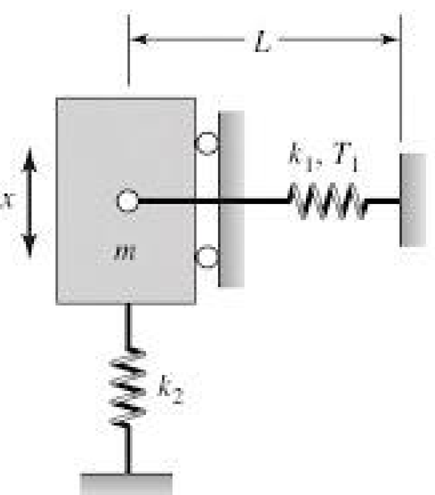

FIGURE 3.12
Single degree-of-freedom system with the horizontal spring under an initial tension T_1.

are considering oscillations about the static-equilibrium position. On substituting for V_1 and V_2 in Eq. (c), we arrive at

$$V(x) = \frac{1}{2}k_1(\delta_o + \Delta L)^2 + \frac{1}{2}k_2x^2 \tag{d}$$

where ΔL is the change in the length of the spring with stiffness k_1 due to the motion x of the mass. For $|x/L| << 1$, as discussed in Example 2.8, this change is

$$\begin{aligned}\Delta L &= \sqrt{L^2 + x^2} - L = L\sqrt{1 + (x/L)^2} - L \\ &\approx L\left(1 + \frac{1}{2}\left(\frac{x}{L}\right)^2\right) - L = \frac{L}{2}\left(\frac{x}{L}\right)^2\end{aligned} \tag{e}$$

From Eqs. (d) and (e), the system potential energy is

$$V(x) = \frac{1}{2}k_1\left(\delta_o + \frac{L}{2}\left(\frac{x}{L}\right)^2\right)^2 + \frac{1}{2}k_2x^2 \tag{f}$$

The expression for potential energy contains terms up to the fourth power of the displacement x, whereas the standard form given in Eqs. (3.46) contains only a quadratic term. However, the kinetic energy is of the form given in Eqs. (3.46). Hence, we will need to use Eq. (3.44) directly to obtain the governing equation. To this end, we recognize that $q_1 = x$ and find that

$$\begin{aligned}\frac{\partial V}{\partial x} &= k_1\left(\delta_o + \frac{L}{2}\left(\frac{x}{L}\right)^2\right)\left(\frac{x}{L}\right) + k_2x \\ &= \left(k_2 + \frac{k_1\delta_o}{L}\right)x + \frac{k_1}{2}\frac{x^3}{L^2} \\ &\cong \left(k_2 + \frac{T_1}{L}\right)x\end{aligned} \tag{g}$$

where we have made use of Eq. (a) and we have dropped the cubic term in x since we have assumed that $|x/L| << 1$.

Noting that the dissipation function $D = 0$ and that the generalized force $Q_1 = 0$, we substitute Eqs. (b) and (g) into Eq. (3.44) to obtain the following governing equation of motion

$$m\ddot{x} + \left(k_2 + \frac{T_1}{L}\right)x = 0 \tag{h}$$

From Eq. (h), we recognize the natural frequency to be

$$\omega_n = \sqrt{\frac{k_2 + T_1/L}{m}} \tag{i}$$

It is seen that the effect of a spring under tension, which is initially normal to the direction of motion, is to increase the natural frequency of the system.

If the spring of constant k_1 is compressed instead of being in tension, then we can replace T_1 by $-T_1$ and Eq. (i) becomes

$$\omega_n = \sqrt{\frac{k_2 - T_1/L}{m}} \tag{j}$$

From Eq. (j), it is seen that the natural frequency can be made very low by adjusting the compression of the spring with stiffness k_1. At the same time, the spring with stiffness k_2 can be made stiff enough so that the static displacement of the system is not excessive. This type of system is the basis of at least one commercial product.[14]

EXAMPLE 3.14 Equation of motion for a disk with an extended mass

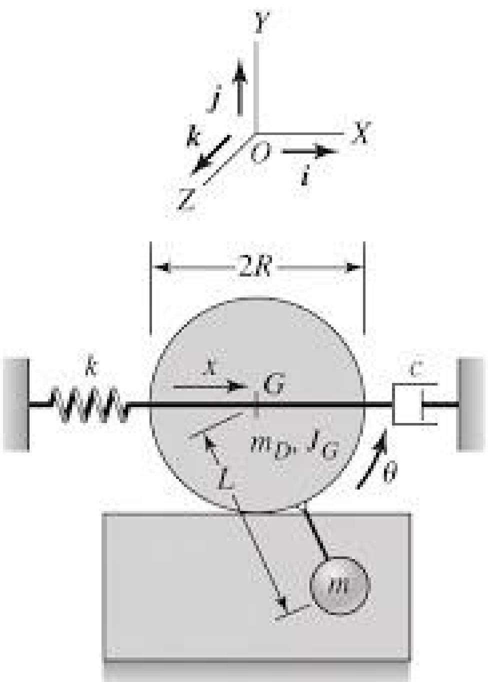

FIGURE 3.13
Disk that is rolling and translating and has a rigidly attached extended mass.

We shall determine the governing equation of motion and the natural frequency for the system shown in Figure 3.13, for "small" angular motions of the pendulum. The system shown in Figure 3.13 is similar to the system shown in Figure 3.9, except that there is an additional pendulum of length L and rigid mass m that is attached to the disk. The disk rolls without slipping. The position of fixed point O is chosen to coincide with the unstretched length of the spring, the coordinate θ is chosen as the generalized coordinate, and the translation $x = -R\theta$.

The kinetic energy of the system is given by

$$T = T_{\text{disk}} + T_{\text{pendulum}} \tag{a}$$

where the kinetic energy of the disk is given by Eq. (e) of Example 3.10. The kinetic energy of the pendulum mass m is given by

$$T_{\text{pendulum}} = \frac{1}{2} m \,(\boldsymbol{V}_m \cdot \boldsymbol{V}_m) \tag{b}$$

where, based on the particle kinematics discussed in Section 1.2 and the first of Eqs. (f) of Example 1.1, we have

$$\boldsymbol{V}_m = \frac{d\boldsymbol{r}_m}{dt} = \frac{d}{dt}[(x + L\sin\theta)\boldsymbol{i} + (L - L\cos\theta)\boldsymbol{j}]$$

$$\boldsymbol{V}_m = (-R\dot{\theta} + L\dot{\theta}\cos\theta)\boldsymbol{i} + L\dot{\theta}\sin\theta\boldsymbol{j} \tag{c}$$

On substituting for the velocity vector from Eq. (c) into Eq. (b) and executing the scalar dot product, we obtain

$$\begin{aligned} T_{\text{pendulum}} &= \frac{1}{2} m\,[(-R\dot{\theta} + L\dot{\theta}\cos\theta)^2 + L^2\dot{\theta}^2\sin^2\theta] \\ &= \frac{1}{2} m\,[R^2\dot{\theta}^2 + L^2\dot{\theta}^2 - 2LR\dot{\theta}^2\cos\theta] \\ &= \frac{1}{2} m\,[(R^2 + L^2 - 2LR\cos\theta)]\dot{\theta}^2 \end{aligned} \tag{d}$$

[14] Minus K Technology, 420 S. Hindry Ave., Unit E, Inglewood, CA, 90301 (www.minusk.com).

Since the objective is to obtain the governing equation for "small" angular oscillations of the pendulum about the position $\theta = 0$, we retain up to quadratic terms in Eq. (d). To this end, we expand the $\cos\theta$ term as

$$\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \cdots \tag{e}$$

substitute Eq. (e) into Eq. (d), and retain up to quadratic terms to obtain

$$T_{\text{pendulum}} = \frac{1}{2} m\,[L^2 + R^2 - 2LR]\dot{\theta}^2 = \frac{1}{2} m(L - R)^2\dot{\theta}^2 \tag{f}$$

Making use of Eq. (e) of Example (3.10) and Eq. (f), we construct the system kinetic energy from Eq. (a) as

$$\begin{aligned} T &= \frac{1}{2} m(L - R)^2\dot{\theta}^2 + \frac{1}{2} m_D\dot{x}^2 + \frac{1}{2} J_G\dot{\theta}^2 \\ &= \frac{1}{2}[m(L - R)^2 + m_D R^2 + J_G]\dot{\theta}^2 \end{aligned} \tag{g}$$

The potential energy of the system is constructed as

$$V = \frac{1}{2} kx^2 + mg(L - L\cos\theta) = \frac{1}{2} kR^2\theta^2 + mgL(1 - \cos\theta) \tag{h}$$

where the datum for the potential energy of the pendulum is located at the bottom position and we have used Eq. (2.36) with $L/2$ replaced by L. To describe small oscillations of the pendulum, we use the expansion for the $\cos\theta$ term given by Eq. (e) and retain up to quadratic terms in Eq. (h) to obtain

$$\begin{aligned} V &= \frac{1}{2} kR^2\theta^2 + \frac{1}{2} mgL\theta^2 \\ &= \frac{1}{2}(kR^2 + mgL)\theta^2 \end{aligned} \tag{i}$$

In this case, the dissipation function is given by

$$D = \frac{1}{2} c\dot{x}^2 = \frac{1}{2} cR^2\dot{\theta}^2 \tag{j}$$

Comparing Eqs. (g), (i), and (j) to Eqs. (3.46), we find that the equivalent system properties are given by

$$\begin{aligned} m_e &= m(L - R)^2 + m_D R^2 + J_G \\ k_e &= kR^2 + mgL \\ c_e &= cR^2 \end{aligned} \tag{k}$$

Thus, making use of the last of Eqs. (3.47) and Eqs. (k) and noting that the generalized force $Q_1 = 0$, we arrive at the governing equation

$$m_e\ddot{\theta} + c_e\dot{\theta} + k_e\theta = 0 \tag{l}$$

From Eqs. (k) and the first of Eqs. (3.48), we determine that the system natural frequency is

$$\omega_n = \sqrt{\frac{k_e}{m_e}} = \sqrt{\frac{kR^2 + mgL}{m(L - R)^2 + m_D R^2 + J_G}} \tag{m}$$

EXAMPLE 3.15 Lagrange formulation for a microelectromechanical system (MEMS) device

We shall determine the governing equation of motion and the natural frequency for the microelectromechanical system[15] shown in Figure 3.14. The mass m_2 is the scanning micro mirror whose typical dimensions are 300 μm × 400 μm. This mass is modeled as a rigid bar. The torsion springs are rods that are 50 μm in length and 4 μm^2 in area and are collectively modeled by an equivalent torsion spring of stiffness k_t in the figure. Mass m_1 is the mass of the electrostatic comb drive, which is comprised of 100 interlaced "fingers." The comb fingers are 2 μm wide and 40 μm long. The comb drive is connected to the displacement drive through an elastic member that has a spring constant k. The mass m_1 is connected to the bar m_2 by a rigid, weightless rod.

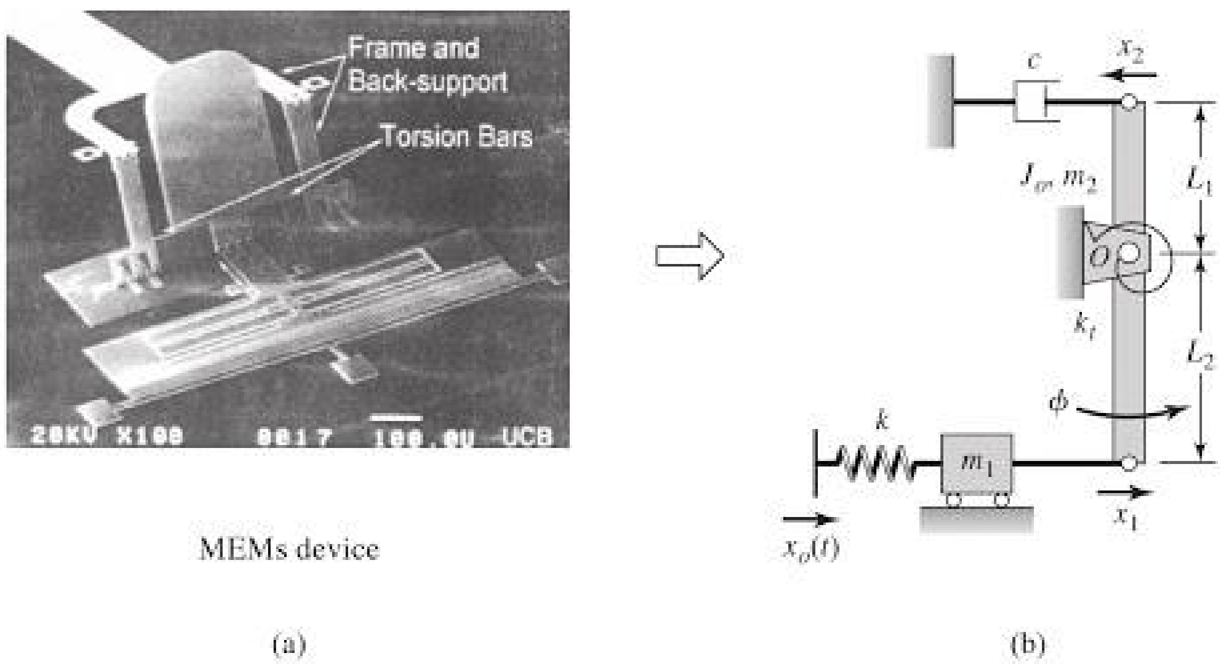

FIGURE 3.14
(a) MEMS device and (b) single degree-of-freedom model. *Source:* From M.H.Kiang, O.Solgaard, K.Y.Lau, and R.S.Muller, "Electrostatic Comb-Drive-Actuated Micromirrors for Laser-Beam Scanning and Positioning", *Journal of Microelectromechanical System*, Vol. 7, No.1, pp. 27–37 (March 1998). Copyright © 1998 IEEE. Reprinted with permission.

[15]M.-H. Kiang, O. Solgaard, K. Y. Lau, and R. S. Muller, "Electrostatic Comb-Drive-Actuated Micromirrors for Laser-Beam Scanning and Positioning," *J. Microelectromechanical Systems*, Vol. 7, No. 1, pp. 27–37 (March 1998).

We will use the angular coordinate ϕ as the generalized coordinate, and derive the equation of motion for "small" angular oscillations. The translation $x_o(t)$ is prescribed, and the translations x_1 and x_2 are approximated as

$$\begin{aligned} x_1 &= L_2\phi \\ x_2 &= L_1\phi \end{aligned} \tag{a}$$

The system potential energy is constructed as

$$V = V_1 + V_2 + V_3 \tag{b}$$

where V_1 is the potential energy of the torsion spring, V_2 is the potential energy of the translation spring, and V_3 is the gravitational potential energy of the bar. For "small" angular oscillations of the bar, Eq. (c) of Example 2.9 is used to describe the bar's potential energy. Thus, we arrive at

$$\begin{aligned} V &= \frac{1}{2}k_t\phi^2 + \frac{1}{2}k(x_o(t) - x_1)^2 + \frac{1}{4}m_2g(L_2 - L_1)\phi^2 \\ &= \frac{1}{2}k_t\phi^2 + \frac{1}{2}k(x_o(t) - L_2\phi)^2 + \frac{1}{4}m_2g(L_2 - L_1)\phi^2 \end{aligned} \tag{c}$$

where we have made use of Eqs. (a). When $L_2 = L_1$, the effects of the increase and decrease in the potential energy of each portion of the bar of mass m_2 cancel.

Next, the system's kinetic energy is determined as

$$\begin{aligned} T &= \frac{1}{2}J_o\dot{\phi}^2 + \frac{1}{2}m_1\dot{x}_1^2 = \frac{1}{2}J_o\dot{\phi}^2 + \frac{1}{2}m_1L_2^2\dot{\phi}^2 \\ &= \frac{1}{2}(J_o + m_1L_2^2)\dot{\phi}^2 \end{aligned} \tag{d}$$

The system dissipation function is given by

$$D = \frac{1}{2}c\dot{x}_2^2 = \frac{1}{2}cL_1^2\dot{\phi}^2 \tag{e}$$

where we have again made use of Eqs. (a). Comparing the forms of Eqs. (c), (d), and (e) to Eqs. (3.46), we find that the potential energy is not in the standard form. Thus, we will make use of Eq. (3.44) to determine the governing equation of motion. To this end, we find from Eq. (c) that

$$\begin{aligned} \frac{\partial V}{\partial \phi} &= \frac{\partial}{\partial \phi}\left[\frac{1}{2}k_t\phi^2 + \frac{1}{2}k\,(x_o(t) - L_2\phi)^2 + \frac{1}{4}m_2g(L_2 - L_1)\phi^2\right] \\ &= k_t\phi - kL_2\,(x_o(t) - L_2\phi) + \frac{1}{2}m_2g\,(L_2 - L_1)\phi \\ &= \left[k_t + kL_2^2 + \frac{1}{2}m_2g\,(L_2 - L_1)\right]\phi - kL_2x_o(t) \end{aligned} \tag{f}$$

To obtain the governing equation of motion, we recognize that $q_1 = \phi$, substitute for the system kinetic energy and the dissipation function from where

$$m(\varphi) = J_{mb} + J_{me} + (J_{mt} + m_s r^2)\left(\frac{ab\cos\varphi - b^2}{r^2(\varphi)}\right)^2 + m_s\left(\frac{ab}{r(\varphi)}\sin\varphi\right)^2 \tag{h}$$

System Potential Energy

The system potential energy is given by

$$V = \frac{1}{2}kr^2(\varphi) + \frac{1}{2}k_d\left[d(t) - e\varphi\right]^2 \tag{i}$$

Equation of Motion

Since the expressions for kinetic energy and potential energy are not in the standard form of Eqs. (3.46), we will make use of the Lagrange equation given by Eq. (3.44) to obtain the equation of motion; that is,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\varphi}\right) - \frac{\partial T}{\partial \varphi} + \frac{\partial D}{\partial \dot\varphi} + \frac{\partial V}{\partial \varphi} = 0 \tag{j}$$

where we have used the fact that the generalized force is zero. Noting that there is no dissipation in the system—that is, $D = 0$—we substitute for the kinetic energy and potential energy from Eqs. (g) and (i), respectively, into Eq. (j), and carry out the differentiation operations to obtain the following nonlinear equation

$$m(\varphi)\ddot\varphi + \frac{1}{2}m'(\varphi)\dot\varphi^2 + kr(\varphi)r'(\varphi) + k_d e^2\varphi = k_d e d(t) \tag{k}$$

where the prime denotes the derivative with respect to φ.

EXAMPLE 3.17 Oscillations of a crankshaft[17]

Consider the model of a crankshaft shown in Figure 3.16 where gravity is acting in the $\boldsymbol{k}$ direction. The crank of mass m_G and mass moment of inertia J_G about its center of mass is connected to a slider of mass m_p at one end and to a disk of mass moment of inertia J_d about the fixed point O. Choosing the angle θ as the generalized coordinate, we will first derive the governing equation

[17]G. Genta, *Vibration of Structures and Machines: Practical Aspects,* 2nd ed., Springer-Verlag, NY, pp. 338–341 (1995); and E. Brusa, C. Delprete, and G. Genta, "Torsional Vibration of Crankshafts: Effects of Non-Constant Moments of Inertia," *J. Sound Vibration,* Vol. 205, No. 2, pp. 135–150 (1997).

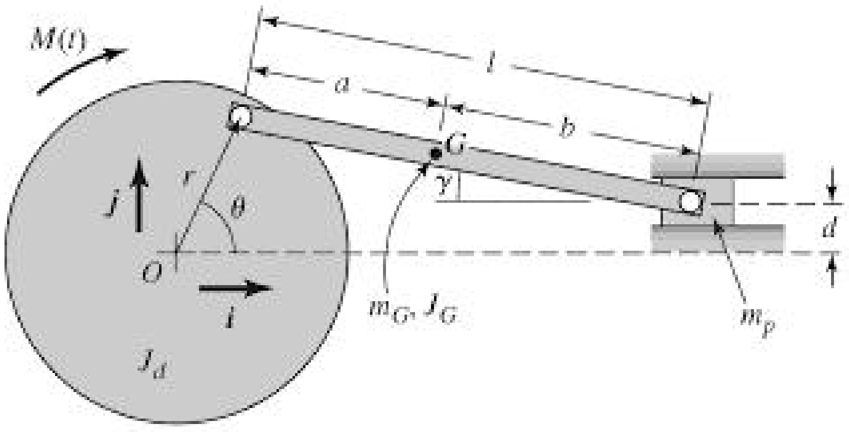

FIGURE 3.16
Crankshaft model.

of motion of the system, and then from this equation, determine the equation governing oscillations about a steady rotation rate.

Kinematics

From Figure 3.16, we see that the position vector of the slider mass m_p with respect to point O is

$$\boldsymbol{r}_p = (r\cos\theta + l\cos\gamma)\boldsymbol{i} + d\boldsymbol{j} \tag{a}$$

and that the position vector of the center of mass G of the crank with respect to point O is

$$\boldsymbol{r}_G = (r\cos\theta + a\cos\gamma)\boldsymbol{i} + (r\sin\theta - a\sin\gamma)\boldsymbol{j} \tag{b}$$

Furthermore, from geometry, the angle γ and the angle θ are related by the relation

$$r\sin\theta = d + l\sin\gamma \tag{c}$$

To determine the slider velocity, we differentiate the position vector $\boldsymbol{r}_p$ with respect to time and obtain

$$\boldsymbol{v}_p = (-r\dot{\theta}\sin\theta - l\dot{\gamma}\sin\gamma)\boldsymbol{i} \tag{d}$$

By differentiating Eq. (c) with respect to time, we obtain the following relationship between $\dot{\gamma}$ and $\dot{\theta}$:

$$\dot{\gamma} = \frac{r\cos\theta}{l\cos\gamma}\dot{\theta} \tag{e}$$

After substituting Eq. (e) into Eq. (d), we obtain the slider velocity to be

$$\boldsymbol{v}_p = -r\dot{\theta}(\sin\theta + \tan\gamma\cos\theta)\boldsymbol{i} \tag{f}$$

The velocity of the center of mass G of the crank is obtained in a similar manner. We differentiate Eq. (b) with respect to time to obtain

$$\boldsymbol{v}_G = (-r\dot{\theta}\sin\theta - a\dot{\gamma}\sin\gamma)\boldsymbol{i} + (r\dot{\theta}\cos\theta - a\dot{\gamma}\cos\gamma)\boldsymbol{j} \tag{g}$$

After substituting Eq. (e) into Eq. (g) and noting that $a + b = l$, we obtain the velocity of the crank's center of mass to be

$$\boldsymbol{v}_G = -\left(\sin\theta + \frac{a}{l}\tan\gamma\cos\theta\right)r\dot{\theta}\boldsymbol{i} + \left(\frac{b}{l}\cos\theta\right)r\dot{\theta}\boldsymbol{j} \tag{h}$$

System Kinetic Energy

The total kinetic energy of the system is given by

$$T = \frac{1}{2}J_d\dot{\theta}^2 + \frac{1}{2}m_G(\boldsymbol{v}_G\cdot\boldsymbol{v}_G) + \frac{1}{2}J_G\dot{\gamma}^2 + \frac{1}{2}m_p(\boldsymbol{v}_p\cdot\boldsymbol{v}_p) \tag{i}$$

We now substitute Eqs. (e), (f), and (h) into Eq. (i) to obtain

$$T = \frac{1}{2}J(\theta)\dot{\theta}^2 \tag{j}$$

where

$$\begin{aligned} J(\theta) = J_d &+ r^2 m_G\left\{\left(\sin\theta + \frac{a}{l}\tan\gamma\cos\theta\right)^2 + \left(\frac{b}{l}\cos\theta\right)^2\right\} \\ &+ J_G\left(\frac{r}{l}\frac{\cos\theta}{\cos\gamma}\right)^2 + r^2 m_p(\sin\theta + \tan\gamma\cos\theta)^2 \end{aligned} \tag{k}$$

and from Eq. (c)

$$\gamma = \sin^{-1}\left\{\frac{r}{l}\sin\theta - \frac{d}{l}\right\} \tag{l}$$

Equation of Motion

Noting that the generalized coordinate $q_1 = \varphi$, the system potential energy is zero, the system dissipation function is zero, and that the generalized moment $Q_1 = -M(t)$, Eq. (3.44) takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = -M(t) \tag{m}$$

Upon substituting Eq. (j) into Eq. (m) and performing the differentiation operations, we obtain

$$J(\theta)\ddot{\theta} + \frac{1}{2}J'(\theta)\dot{\theta}^2 = -M(t) \tag{n}$$

where the prime denotes the derivative with respect to θ.

The angle θ can be expressed as the superposition of a rigid-body motion at a constant angular velocity ω and an oscillatory rotation ϕ; that is,

$$\theta(t) = \omega t + \phi(t) \tag{o}$$

Then, from Eqs. (n) and (o), we arrive at

$$J(\theta)\ddot{\phi} + \frac{1}{2}J'(\theta)(\omega + \dot{\phi})^2 = -M(t) \tag{p}$$

EXAMPLE 3.18 Vibration of a centrifugal governor[18]

A centrifugal governor is a device that automatically controls the speed of an engine and prevents engines from exceeding certain speeds or prevents damage from sudden changes in torque loading. We shall derive the equation of motion of such a governor by using Lagrange's equation. A model of this device is shown in Figure 3.17.

The velocity vector relative to point o of the left hand mass is given by

$$\boldsymbol{V}_m = -L\dot{\varphi}\cos\varphi\boldsymbol{i} + L\dot{\varphi}\sin\varphi\boldsymbol{j} + (r + L\sin\varphi)\omega\boldsymbol{k} \tag{a}$$

From Eq. (1.22) and Eq. (a), the kinetic energy is

$$\begin{aligned} T(\varphi, \dot{\varphi}) &= 2\left[\frac{1}{2}m(\boldsymbol{V}_m \cdot \boldsymbol{V}_m)\right] \\ &= m\left[(-L\dot{\varphi}\cos\varphi)^2 + (L\dot{\varphi}\sin\varphi)^2 + ((r + L\sin\varphi)\omega)^2\right] \\ &= m\omega^2(r + L\sin\varphi)^2 + m\dot{\varphi}^2L^2 \end{aligned} \tag{b}$$

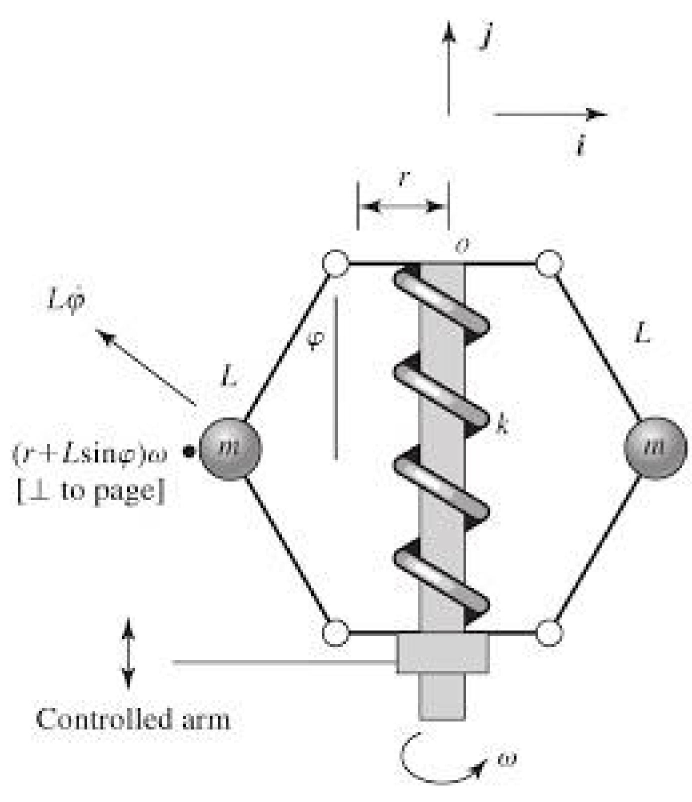

FIGURE 3.17
Centrifugal governor.

[18] J. P. Den Hartog, *Mechanical Vibrations*, Dover, p. 309, 1985; and Z.-M. Ge and C.-I Lee, "Nonlinear Dynamics and Control of Chaos for a Rotational Machine with a Hexagonal Centrifugal Governor with a Spring," *J. Sound Vibration*, 262, pp. 845–864, 2003.

The potential energy with respect to the static equilibrium position is

$$V(\varphi) = \frac{1}{2}k\,(2L\,(1 - \cos\varphi))^2 - 2mgL\cos\varphi \tag{c}$$

where the factor of 2 inside the parenthesis is because each pair of linkages compresses the spring from both the top and the bottom.

Upon using Eq. (3.44) with $q_1 = \varphi$, noting that $Q_1 = D = 0$, and performing the required operations, we obtain the following governing equation

$$mL^2\ddot{\varphi} - mrL\omega^2\cos\varphi - (m\omega^2 + 2k)L^2\sin\varphi\cos\varphi + L(mg + 2kL)\sin\varphi = 0 \tag{d}$$

Introducing the quantities

$$\gamma = \frac{r}{L}, \qquad \omega_p^2 = \frac{g}{L}, \qquad \text{and} \qquad \omega_n^2 = \frac{2k}{m} \tag{e}$$

Eq. (d) is rewritten as

$$\ddot{\varphi} - \gamma\omega^2\cos\varphi - (\omega^2 + \omega_n^2)\sin\varphi\cos\varphi + (\omega_p^2 + \omega_n^2)\sin\varphi = 0 \tag{f}$$

If we assume that the oscillations φ about $\varphi = 0$ are small, then $\cos\varphi \approx 1$, $\sin\varphi \approx \varphi$, and Eq. (f) simplifies to

$$\ddot{\varphi} + (\omega_p^2 - \omega^2)\varphi = \gamma\omega^2 \tag{g}$$

From the stiffness coefficient in the equation of motion, we see that for $\omega > \omega_p$ the stiffness coefficient is negative.

EXAMPLE 3.19 Oscillations of a rotating system

A cylindrical wheel is placed on a platform that is rotating about its axis with an angular speed Ω. The center of the wheel is attached to the platform by a spring with constant k, as shown in Figure 3.18. We shall determine the change in the equilibrium position of the wheel and the natural frequency of the system about this equilibrium position. When $\Omega = 0$, the center of the wheel is at a distance R from the axis of rotation, which is the length of the unstretched spring.

If we denote the change in the equilibrium of the spring due to the rotation Ω as δ, then at equilibrium, the spring force is equal to the centrifugal force, which can be represented as

$$k\delta = m(R + \delta)\Omega^2$$

inertia of the bar about the point O is J_O, and the torsion stiffness of the spring attached to the pivot point is k_t. Assume that there is gravity loading.

3.8 Determine the equation governing the system studied in Example 3.15 by carrying out a force balance.

Section 3.3.1

3.9 A cylindrical buoy with a radius of 1.5 m and a mass of 1000 kg floats in salt water ($\rho = 1026$ kg/m^3). Determine the natural frequency of this system.

3.10 A 10 kg instrument is to be mounted at the end of a cantilever arm of annular cross section. The arm has a Young's modulus of elasticity $E = 72 \times 10^9$ N/m^2 and a mass density $\rho = 2800$ kg/m^3. If this arm is 500 mm long, determine the cross-section dimensions of the arm so that the first natural frequency of the system is above 50 Hz.

3.11 The static displacement of a system with a motor weight of 385.6 kg is found to be 0.0254 mm. Determine the natural frequency of vertical vibrations of this system.

3.12 A rotor is attached to one end of a shaft that is fixed at the other end. Let the rotary inertia of the rotor be J_G, and assume that the rotary inertia of the shaft is negligible compared to that of the rotor. The shaft has a diameter d, a length L, and it is made from material with a shear modulus G. Determine an expression for the natural frequency of torsional oscillations.

3.13 Obtain an expression for the natural frequency for the system shown in Figure E3.5.

3.14 Consider the hand motion discussed in Example 3.3 and let the hand move in the horizontal plane; that is, the gravity force acts normal to this plane. Assume that the length of the forearm l is 25 cm, the mass of the fore arm m is 1.5 kg, the object being carried in the hand has a mass $M = 5$ kg, the constant k_b associated with the restoring force of the biceps is 2×10^3 N/rad, the constant K_t associated with the triceps is 2×10^3 N/rad, and the spacing $a = 4$ cm. Determine the equation of motion of this system, and from this governing equation, find the natural frequency and damping factor of the system.

3.15 A spring elongates 2.5 mm when stretched by a force of 5 N. Determine the static deflection and the period of vibration if a mass of 8 kg is attached to the spring.

3.16 Determine the natural frequency of the steel disk with torsion spring shown in Figure E3.16 when $k_t = 0.488$ N·m/rad, $d = 50$ mm, $\rho_{disk} = 7850$ kg/m^3, and $h = 2$ mm.

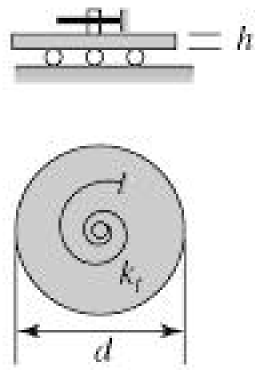

FIGURE E3.16

3.17 Consider a nonlinear spring that is governed by the force-displacement relationship

$$F(x) = a\left(\frac{x}{b}\right)^c$$

where $a = 3000$ N, $b = 0.015$ m, and $c = 2.80$. If this spring is to be used as a mounting for different machinery systems, obtain a graph similar to that shown in Figure 3.5b and discuss how the natural frequency of this system changes with the weight of the machinery.

3.18 The static deflection in the tibia bone of a 120 kg person standing upright is found to be 25 μm. Determine the associated natural frequency of axial vibrations.

3.19 A solid wooden cylinder of radius r, height h, and specific gravity s_w is placed in a container of tap water such that the axis of the cylinder is perpendicular to the surface of the water. Assume that the density of the water is ρ_{H_2O}. It is assumed the wooden cylinder stays upright under small oscillations.

a) If the cylinder is displaced a small amount, then determine an expression for its natural frequency.
b) If the tap water is replaced by salt water with specific gravity of 1.2, then determine whether the natural frequency of the wooden cylinder increases or decreases and by what percentage.

3.20 Consider the pulley system shown in Figure E3.20. The mass of each pulley is small compared with the mass m and, therefore, can be ignored. Furthermore, the cord holding the mass is inextensible and has negligible mass. Obtain an expression for the natural frequency of the system.

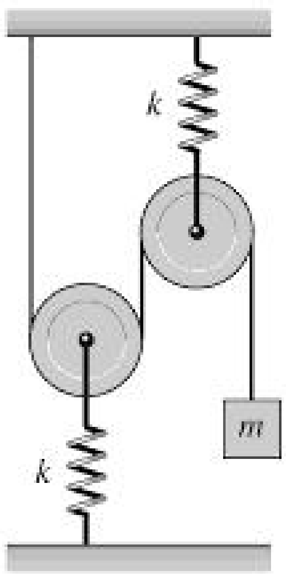

FIGURE E3.20

3.21 A rectangular block of mass m rests on a stationary half-cylinder, as shown in Figure E3.21. Find the natural frequency of the block when it undergoes small oscillations about the point of contact with the cylinder.

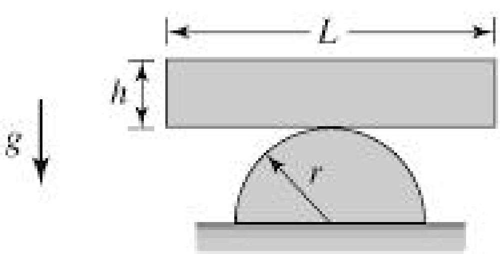

FIGURE E3.21

Section 3.3.2

3.22 Formulate a design guideline for Example 3.8 that would enable a vibratory system designer to decrease the static deflection by a factor n while holding the damping ratio and damping coefficient constant.

3.23 Formulate a design guideline for Example 3.8 that would enable a vibratory system designer to decrease the static deflection by a factor n while keeping the damping ratio and mass m constant.

3.24 An instrument's needle indicator has a rotary inertia of 1.4×10^6 kg·m^2. It is attached to a torsion spring whose stiffness is 1.1×10^{-5} N·m/rad and a viscous damper of coefficient c. What is the value of c needed so that the needle is critically damped?

3.25 Determine the natural frequency and damping factor for the system shown in Figure E2.26.

3.26 Determine the natural frequency and damping factor for the system shown in Figure E2.27.

Section 3.6

3.27 For the base-excitation prototype shown in Figure 3.6, assume that the base displacement $y(t)$ is known, choose $x(t)$ as the generalized coordinate, and derive the equation of motion by using Lagrange's equation.

3.28 Obtain the equation of motion for the system with rotating unbalance shown in Figure 3.7 by using Lagrange's equations.

3.29 Obtain the equation of motion for the system shown in Figure 3.10 by using moment balance and compare it to the results obtained by using Lagrange's equation.

3.30 Derive the governing equation for the single-degree-of-freedom system shown in Figure E3.30 in terms of θ when θ is small, and obtain an expression for its natural frequency. The top mass of the pendulum is a sphere, and the mass m_r of the horizontal rod and the mass m_p of the rod that is supporting m_a are

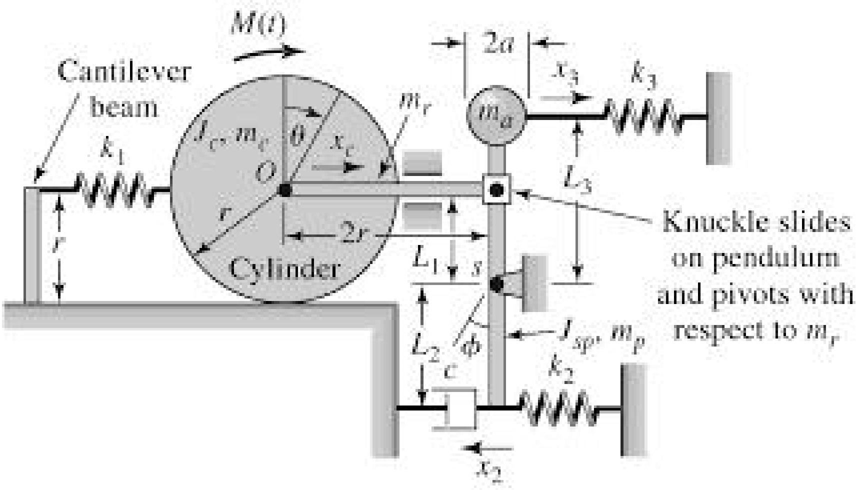

FIGURE E3.30

each uniformly distributed. The cylinder rolls without slipping. The rotational inertia J_c of the cylinder is about the point O and J_{sp} is the total rotational inertia of the rod about the point s. Assume that these rotational inertias are known.

3.31 For the fluid-float system shown in Figure E3.31, J_o is the mass moment of inertia about point O. Assume that the mass of the bar is m_b. Answer the following.

a) For "small" angular oscillations, derive the governing equation of motion for the fluid float system.
b) What is the value of the damping coefficient c for which the system is critically damped?

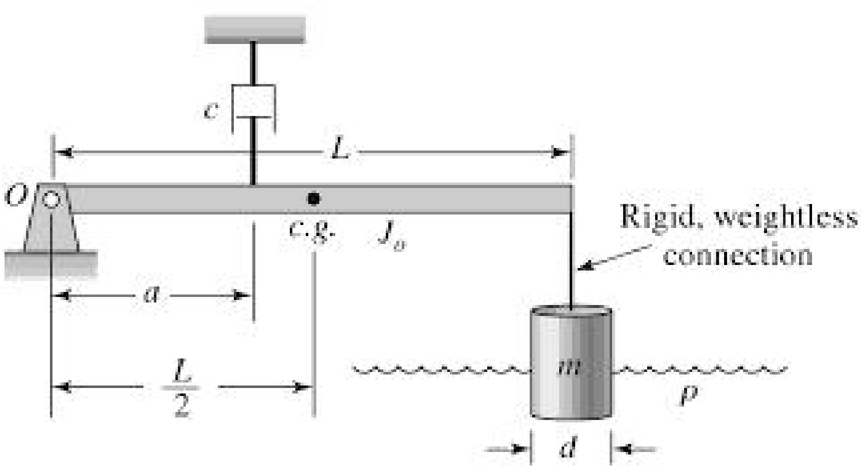

FIGURE E3.31

3.32 Determine the nonlinear governing equation of motion for the kinematically constrained system shown in Figure E3.32. Consider only vertical motions of m_1.

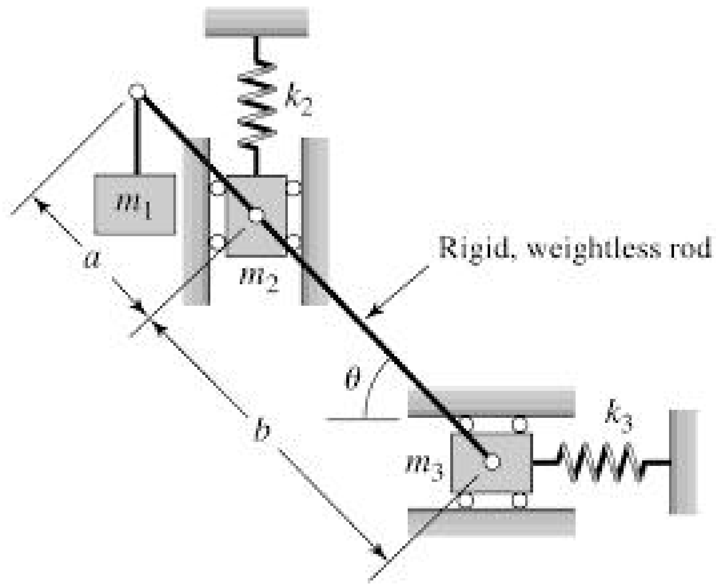

FIGURE E3.32

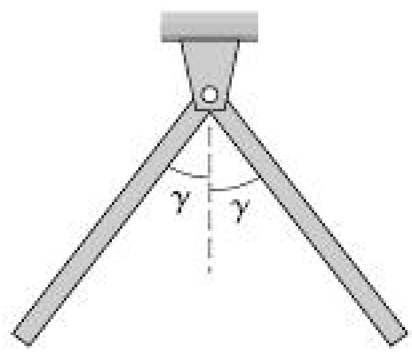

FIGURE E3.33

3.33 Determine the natural frequency of the angle bracket shown in Figure E3.33. Each leg of the bracket has a uniformly distributed mass m and a length L.

3.34 Determine the natural frequency for the vertical oscillations of the system shown in Figure E3.34. Let L be the static equilibrium length of the spring and let $|x/L| << 1$. The angle γ is arbitrary.

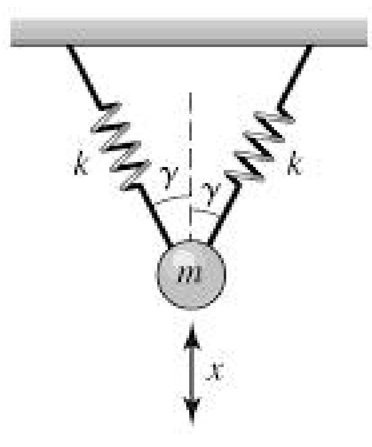

FIGURE E3.34

3.35 Consider the planar pendulum of mass m and constant length l that is shown in Figure E3.35. This pendulum is described by the following nonlinear equation

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

where θ is the angle measured from the vertical. Determine the static-equilibrium positions of this system and linearize the system for "small" oscillations about each of the system static-equilibrium positions.

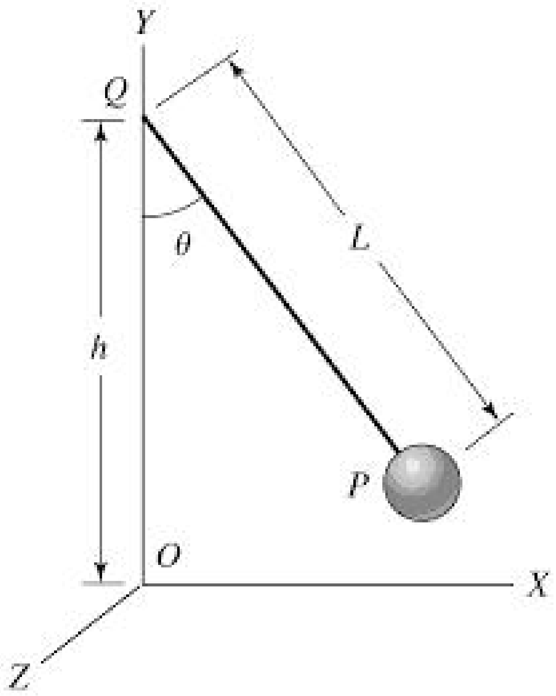

FIGURE E3.35

3.36 For the translating and rotating disc system of Figure 3.9, choose the coordinate x measured from the unstretched length of the spring to describe the motion of the system. What are the equivalent inertia, equivalent stiffness, and equivalent damping properties for this system?

3.37 For the inverted pendulum system of Figure 3.10, choose the coordinate x_1 measured from the unstretched length of the spring to describe the motion of the system. What are the equivalent inertia, equivalent stiffness, and equivalent damping properties for this system?

3.38 Consider a pendulum with an oscillating support as shown in Figure E3.38. The support is oscillating harmonically at a frequency ω; that is,

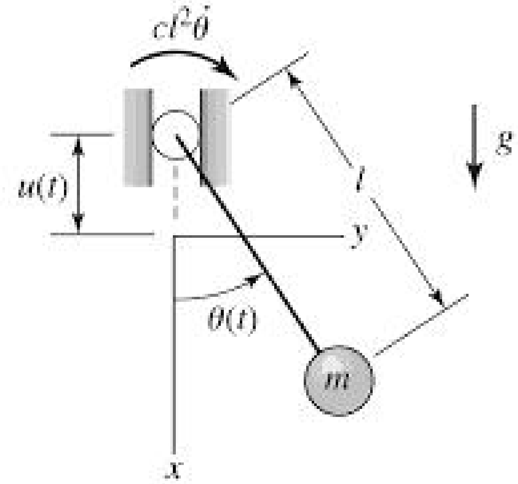

FIGURE E3.38

$$u(t) = U\cos\omega t$$

At the point about which the pendulum rotates, there is a viscous damping moment $cl^2\dot{\theta}$.

a) Determine expressions for the kinetic energy and the potential energy of the system.
b) Show that the governing equation of motion can be written as

$$\frac{d^2\theta}{d\tau^2} + 2\zeta\frac{d\theta}{d\tau} + [1 - U_o\Omega^2\cos\Omega\tau]\sin\theta = 0$$

where

$$\tau = \omega_o t, \quad \omega_o^2 = \frac{g}{l}, \quad \Omega = \frac{\omega}{\omega_o},$$

$$2\zeta = \frac{c}{m\omega_o}, \quad \text{and} \quad U_o = \frac{U}{l}$$

c) Approximate the governing equation in (b) for "small" angular oscillations about $\theta_o = 0$ using a two-term Taylor expansion for $\sin\theta$, and show that the nonlinear stiffness is of the softening type.

3.39 Use Lagrange's equation to derive the equation describing the vibratory system shown in Figure E3.39, which consists of two gears, each of radius r and rotary inertia J. They drive an elastically constrained rack of mass m. The elasticity of the constraint is k. From the equation of motion, determine an expression for the natural frequency.

3.40 Obtain the governing equation of motion in terms of the generalized coordinate θ for torsional

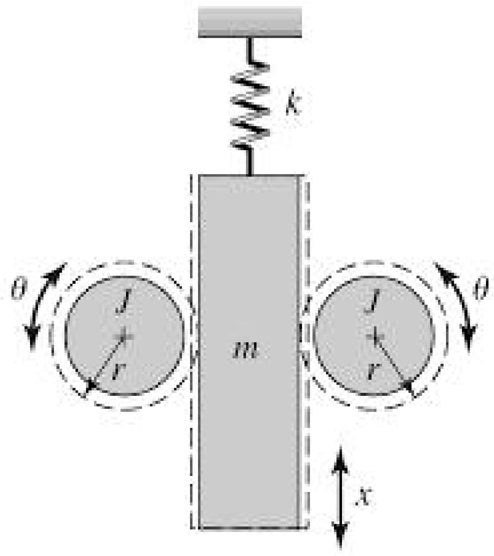

FIGURE E3.39

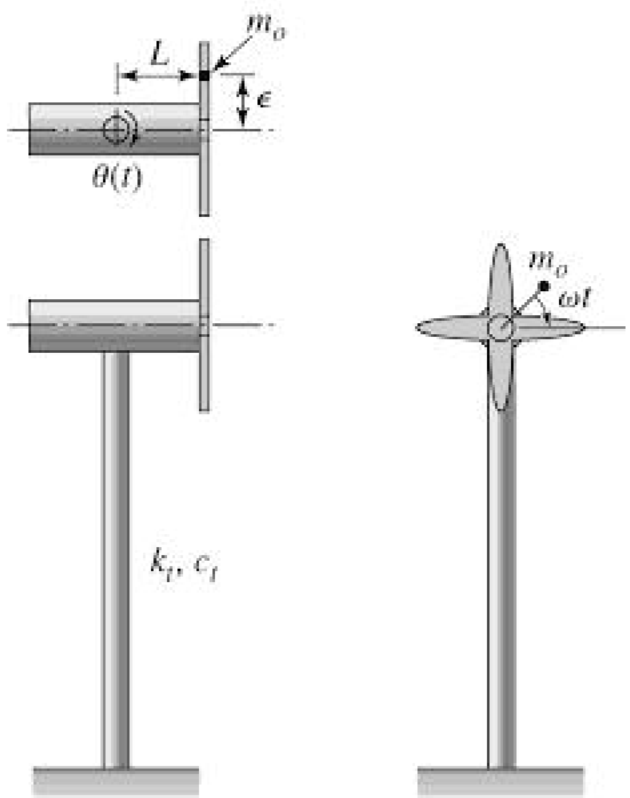

FIGURE E3.40

oscillations of the wind turbine shown in Figure E3.40. Assume that the turbine blades spin at ω rad/s and that the total mass unbalance is represented by mass m_o located at a distance ε from the axis of rotation. The support for the turbine is a solid circular rod of diameter d, length L, and it is made from a material with a shear modulus G. The turbine body and blades have a rotary inertia J_z. Assume that the damping coefficient for torsional oscillations is c_t.

3.41 The uniform concentric cylinder of radius R rolls without slipping on the inclined surface as shown in Figure E3.41. The cylinder has another cylinder of radius $r < R$ concentrically attached to it. The smaller cylinder has a cable wrapped around it. The other end of the cable is fixed. The cable is parallel to the inclined surface. If the stiffness of the cable is k, the mass and rotary inertia of the two attached cylinders are m and J_O, respectively, then determine an expression for the natural frequency of the system in Hz. The length of the unstretched spring is L.

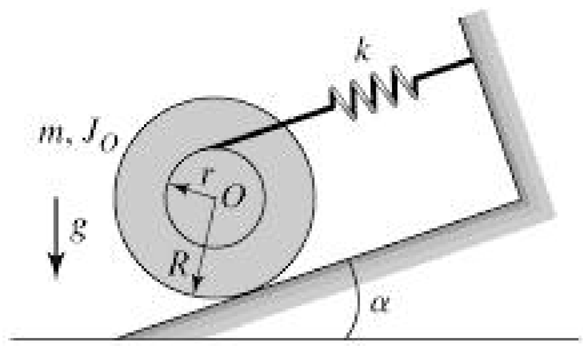

FIGURE E3.41

3.42 For the pulley shown in Figure E3.42, determine an expression for natural frequency, for oscillations about the static equilibrium position. The springs are stretched by an amount x_o at the static equilibrium. The rotary inertia of the pulley about its center is J_O, the radius of the pulley is r, and the stiffness of each translation spring is k.

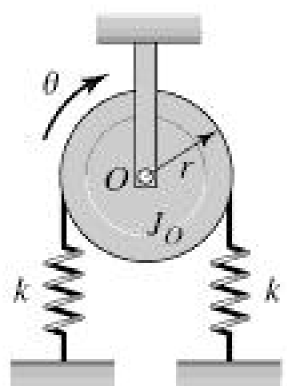

FIGURE E3.42

3.43 The pendulum shown in Figure E3.43 oscillates about the pivot at O. If the mass of the rigid bar of length L_3 can be neglected, then determine an expression for the *damped* natural frequency of the system for "small" angular oscillations.

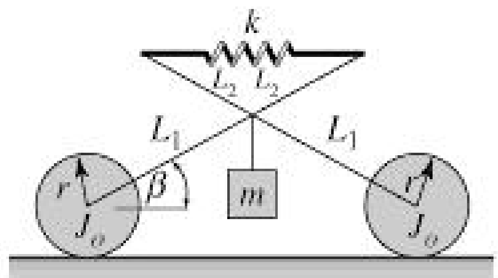

FIGURE E3.52

3.53 Obtain the equation of motion for the system shown in Example 3.12 when the oscillations about its upright position are no longer "small."

3.54 A cylindrical disk of mass m and radius r rolls on a surface without slipping, as shown in Figure E3.54.

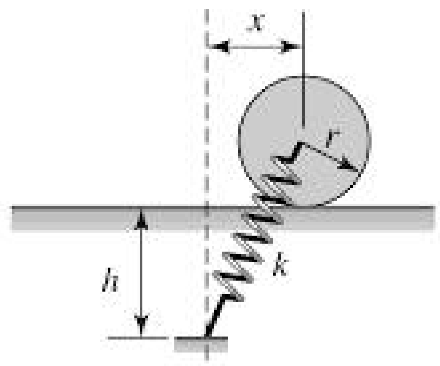

FIGURE E3.54

The free length of the spring L is such that $L = h + r$. Derive the governing equation of motion. Do not make any assumptions about the magnitude of oscillations.

Free oscillations of systems are important considerations that must be taken into account in order to obtain effective operations of a system. For a helicopter or a ship crane, the load oscillations must be taken into account to carry out safe load-transfer operations. Stability of vibratory systems such as the machine tool must also be considered in the design of systems subjected to dynamic loads. (*Source:* David Buttington /Getty Images.)

4

Single Degree-of-Freedom System: Free-Response Characteristics

4.1 INTRODUCTION

In Chapter 3, we illustrated how the governing equation of a single degree-of-freedom system can be derived. In this chapter, the solution of this governing equation is determined, and based on this solution, the responses of single degree-of-freedom systems subjected to different types of initial conditions are discussed. As pointed out in Chapter 3, it is shown that the free responses can be characterized in terms of the damping factor. The notion of stability of a solution is introduced and briefly discussed. The problem of machine-tool chatter during turning operations is also considered and numerical determination of stability for this problem is illustrated. The forced responses of single degree-of-freedom systems are addressed in Chapters 5 and 6.

For all linear single degree-of-freedom systems, the governing equation can be put in the form of Eq. (3.22), which is repeated below.

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = \frac{f(t)}{m} \tag{4.1}$$

A solution is sought for the system described by Eq. (4.1) for a given set of initial conditions. This type of problem is called an *initial-value problem*. Since the system inertia, stiffness, and damping parameters are constant with respect to time, the coefficients in Eq. (4.1) are constant with respect to time. For such linear differential systems with constant coefficients, the solution can be determined by using time-domain methods and the Laplace transform method[1], as illustrated in Appendix D. The latter has been used here, since a general solution for the response of a forced vibratory system can be determined for arbitrary forms of forcing. However, a price that one pays for generality is that in the Laplace transform method the oscillatory characteristics of the vibratory system are not readily apparent until the final solution is determined. On the other hand, when time-domain methods are used, the explicit forms of the solutions assumed in the initial development allows one to readily see the oscillatory characteristics of a vibratory system. In order to provide a flavor of this complementary approach, time-domain methods are summarized in Appendix D.

The ease with which we can use Laplace transforms to solve linear, ordinary differential equations is illustrated by solving for the response of a system with a Maxwell material later in the chapter and by solving for the response of a two degree-of-freedom system in Chapter 8. We also show how to use Laplace transforms to solve for the free responses of thin beams in Chapter 9. An advantage of using the Laplace transform approach is the convenience with which one can see the duality of the responses in the time domain and the frequency domain; this is important for understanding how the same information can be expressed in the two different domains.

In this chapter, we shall show how to:

- Determine the solutions for a linear, single degree-of-freedom system that is underdamped, critically damped, overdamped, and undamped.
- Determine the response of single degree-of-freedom systems to initial conditions and use the results to study the response to impact and collision.
- Determine when a system is stable and how to use the root-locus diagram to obtain stability information.
- Obtain the conditions under which a machine tool chatters.
- Use different models for damping: viscous (Voigt), Maxwell, hysteretic.
- Examine systems with nonlinear stiffness and nonlinear damping.

[1]See Appendix A.

4.2 FREE RESPONSES OF UNDAMPED AND DAMPED SYSTEMS

4.2.1 Introduction

In this section, the responses of undamped and damped single degree-of-freedom systems in the absence of forcing—that is, $f(t) = 0$—are explored in detail. These responses are also referred to as *free responses*, and when the system is undamped or underdamped, the responses are referred to as *free oscillations*. In the absence of forcing, the single degree-of-freedom given by Eq. (4.1) reduces to

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = 0 \tag{4.2}$$

Free responses are the responses of a system to either an initial displacement $x(t) = X_o$, an initial velocity $\dot{x}(0) = V_o$, or to both an initial displacement and an initial velocity. Based on the discussion in Appendix D, there are four distinct types of solutions to Eq. (4.1) depending on the magnitude of the damping factor ζ. These four regions describe four different types of systems as follows.

Underdamped System: $0 < \zeta < 1$

When the damping factor is in the range $0 < \zeta < 1$, we denote the system as an *underdamped* system. From Eq. (3.20), we see that in this region, the damping coefficient c is less than the critical damping coefficient c_c. For values of ζ in this range, the solutions to Eq. (4.2) are given by either Eq. (D.15) or Eq. (D.16); that is,

$$x(t) = X_o e^{-\zeta\omega_n t}\cos(\omega_d t) + \frac{V_o + \zeta\omega_n X_o}{\omega_d} e^{-\zeta\omega_n t}\sin(\omega_d t) \tag{4.3}$$

or

$$x(t) = A_o e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi_d) \tag{4.4}$$

respectively, where

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{4.5}$$

where ω_d is the *damped natural frequency* and

$$A_o = \sqrt{X_o^2 + \left(\frac{V_o + \zeta\omega_n X_o}{\omega_d}\right)^2}$$

$$\varphi_d = \tan^{-1}\frac{\omega_d X_o}{V_o + \zeta\omega_n X_o} \tag{4.6}$$

Design Guideline: The free response of a critically damped system reaches its equilibrium or rest position in the shortest possible time.

In the absence of forcing, when $\zeta > 0$, the displacement response always decays to the equilibrium position $x(t) = 0$. However, this is not true when $\zeta < 0$; the response of the system will grow with respect to time. This is an example of an unstable response, which is discussed in Section 4.3.

Next, we present three examples that explore the free responses of underdamped and critically damped systems in detail.

EXAMPLE 4.1 Free response of a microelectromechanical system

A microelectromechanical system has a mass of 0.40 μg, a stiffness of 0.08 N/m, and a negligible damping coefficient. The gravity loading is normal to the direction of motion of this mass. We shall determine and discuss the displacement response of this system when there is no forcing acting on this system and when the initial displacement is 2 μm and the initial velocity is zero.

Since $V_o = f(t) = \zeta = 0$, we see from Eq. (4.10) that the displacement response has the form

$$x(t) = X_o \cos(\omega_n t) \tag{a}$$

where

$$\omega_n = \sqrt{\frac{k}{m}} \tag{b}$$

From Eq. (b), the natural frequency is

$$\omega_n = \sqrt{\frac{0.08 \text{ N/m}}{0.40 \times 10^{-9} \text{ kg}}} = 14142.14 \text{ rad/s}$$

or

$$f_n = \frac{\omega_n}{2\pi} = \frac{14142.14}{2\pi} = 2250.8 \text{ Hz}$$

Substituting this value and the given value of initial displacement 2 μm into Eq. (a) results in

$$x(t) = 2\cos(14142.14t)\ \mu\text{m} \tag{c}$$

Equation (c) is the displacement response. Based on the form of Eq. (a) or Eq. (c), it is clear that the displacement is a cosine harmonic function that varies periodically with time and has the period

$$T = \frac{2\pi}{\omega_n} = \frac{1}{f_n} = \frac{1}{2250.8} = 444.29\ \mu s$$

From the form of Eq. (c), it is clear that the response does not decay, and hence, the response does not settle down to the static-equilibrium position. The system, instead, oscillates harmonically about this equilibrium position with an amplitude of 2 μm.

EXAMPLE 4.2 Free response of a car tire

A wide-base truck tire is characterized with a stiffness of 1.23×10^6 N/m, an undamped natural frequency of 30 Hz, and a damping coefficient of 4400 N·s/m. In the absence of forcing, we shall determine the response of the system assuming non-zero initial conditions, evaluate the damped natural frequency of the system, and discuss the nature of the response.

Let the mass of the tire be represented by m. Based on the equation of motion derived in Chapter 3 for the system shown in Figure 3.1, the governing equation of motion of the tire system from the static equilibrium position is given by Eq. (4.2); that is,

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = 0 \qquad \text{(a)}$$

For this case,

$$\omega_n = 2\pi \times 30 = 188.50 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{c\omega_n}{2k} = \frac{4400 \text{ N}\cdot\text{s/m} \times 188.50 \text{ rad/s}}{2 \times 1.23 \times 10^6 \text{ N/m}} = 0.337 \qquad \text{(b)}$$

Since the damping factor is less than 1, the system is underdamped. Hence, the solution for Eq. (a) is given by Eq. (4.4); that is, the displacement response of the tire system about the static-equilibrium position is

$$x(t) = A_o e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi_d) \qquad \text{(c)}$$

where the constants A_o and φ_d are determined by the initial displacement and initial velocity as indicated by Eqs. (4.6). The damping factor ζ and the natural frequency ω_n are given by Eqs. (b), and the damped natural frequency ω_d is determined from Eq. (4.5) as

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 188.50\sqrt{1-0.337^2} = 177.5 \text{ rad/s}$$

The response given by Eq. (c) has the form of a damped sinusoid with a period

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{177.5} = 0.0354 \text{ s}$$

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Force Transmitted to Fixed Surface

We shall now determine the dynamic component of the force transmitted to the base of a single degree-of-freedom system such as that shown in Figure 3.1. This force is given by Eq. (3.10); that is,

$$F_R = c\dot{x} + kx \tag{4.22}$$

Upon substituting Eqs. (4.13) and (4.14) into Eq. (4.22), we obtain

$$F_R(t) = \frac{kV_o e^{-\zeta\omega_n t}}{\omega_d}[-2\zeta \sin(\omega_d t - \varphi) + \sin(\omega_d t)] \tag{4.23}$$

At $t = 0$, the reaction force acting on the base is determined from Eq. (4.23) to be

$$F_R(0) = \frac{2\zeta kV_o}{\omega_d}\sin(\varphi) = \frac{2\zeta kV_o}{\omega_n} \tag{4.24}$$

Thus, when the mass of a single degree-of-freedom system is subjected to an initial velocity, the force is instantaneously transmitted to the base. This unrealistic characteristic is a property of modeling the system with a spring and viscous damper combination in parallel. The viscous damper essentially "locks" with the sudden application of the velocity and is thereby momentarily rigid. This temporary rigidity shorts the spring and instantaneously transmits the force to the base. Representing a support by a combination of a linear spring and linear viscous damper in parallel is called the *Kelvin-Voigt model,* which is one type of elementary *viscoelastic model.* A second type of elementary viscoelastic model, called the *Maxwell model,* consists of a linear spring and a linear viscous damper in series, and this model is discussed in Example 4.7.

State-Space Plot and Energy Dissipation

The values of the displacements and velocities corresponding to these maxima and minima can also be visualized in a state-space plot, which is a graph of the displacement versus the velocity at each instant of time. This graph for the system considered here is shown in Figure 4.5. As time unfolds, the trajectory initiated from a set of initial conditions is attracted to the equilibrium position located at the origin (0, 0). When $\zeta=0$, the state-space plot in terms of the nondimensional displacement and nondimensional velocity is a circle. If this plot is made in terms of dimensional quantities, it will be an ellipse.

We now show how the energy dissipated by the system in the time interval $0 \le t \le t_{d,\max}$ can be determined. The system of interest is a spring-mass-damper system, as shown in Figure 3.2, which is translating back and forth along the x-axis. The energy dissipated by the system is equal to the difference between the sum of the kinetic energy and the potential energy in the final state and the sum of the kinetic energy and the potential energy in the initial state. Noting that the potential energy in the initial state is zero, and the kinetic energy in the final state is zero, the energy that is dissipated is the difference between the initial kinetic energy and the potential energy that is stored in the

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The significance of these results is illustrated with several examples, where free oscillations of underdamped systems due to impacts are considered.

EXAMPLE 4.4 Impact of a vehicle bumper[2]

Consider a vehicle of mass m that is travelling at a constant velocity V_o as shown in Figure 4.6a. The bumper is modeled as a spring k and viscous damper c in parallel. If the vehicle's bumper hits a stationary barrier, then after the impact, the displacement and velocity of the mass are those given by Eqs. (4.13) and (4.14), respectively. These results are used to determine the coefficient of restitution of the system and the amount of energy that has been dissipated until the time the bumper is no longer in contact with the barrier.

The bumper is in contact with the barrier only while the sum of the forces

$$kx(t) + c\dot{x}(t) > 0$$

that is, while the spring-damper combination is being compressed. At the instant when they are no longer in compression the acceleration is zero; that is, the time at which the sum of these forces on the mass is zero. The first time instance at which the acceleration is zero is given by Eq. (4.20) for $p = 0$, and the corresponding velocity is given by Eq. (4.21).

Based on Newton's law of impact, the coefficient of restitution ϵ is defined as

$$\epsilon = \frac{-(v_{\text{vehicle}} - v_{\text{barrier}})_{\text{after impact}}}{(v_{\text{vehicle}} - v_{\text{barrier}})_{\text{before impact}}} = \frac{-(v_{\text{vehicle}})_{\text{after impact}}}{(v_{\text{vehicle}})_{\text{before impact}}} \tag{a}$$

where v_{vehicle} is the vehicle velocity, v_{barrier} is the velocity of the barrier, and the assumption that v_{barrier} is zero has been used; that is, the barrier is fixed. Then, making use of Eq. (a) and Eq. (4.21) with $p = 0$, we find that

$$\epsilon = \frac{-\dot{x}(t_{vm})}{\dot{x}(0)} = \frac{V_o e^{-2\varphi/\tan\varphi}}{V_o} = e^{-2\varphi/\tan\varphi} \tag{b}$$

We now make use of Eq. (b) to examine how the coefficient of restitution ϵ depends on the damping factor ζ. Considering first the undamped case, we note from Eq. (4.15) that $\varphi/\tan\varphi \to 0$ as $\zeta \to 0$, and, therefore, $\epsilon \to 1$. In other words, there are no losses and the system leaves with the same velocity with which it arrived. This is consistent with the fact that this is an elastic collision. When $\zeta \to 1$, the system becomes critically damped and $\varphi/\tan\varphi \to 1$ and, therefore, from Eq. (b), we find that $\epsilon \to e^{-2}$; that is, the mass leaves the barrier with a velocity of $0.135V_o$.

[2]See also, V. I. Babitsky, *Theory of Vibro-Impact Systems and Applications*, Springer-Verlag, Berlin, Appendix I (1998).

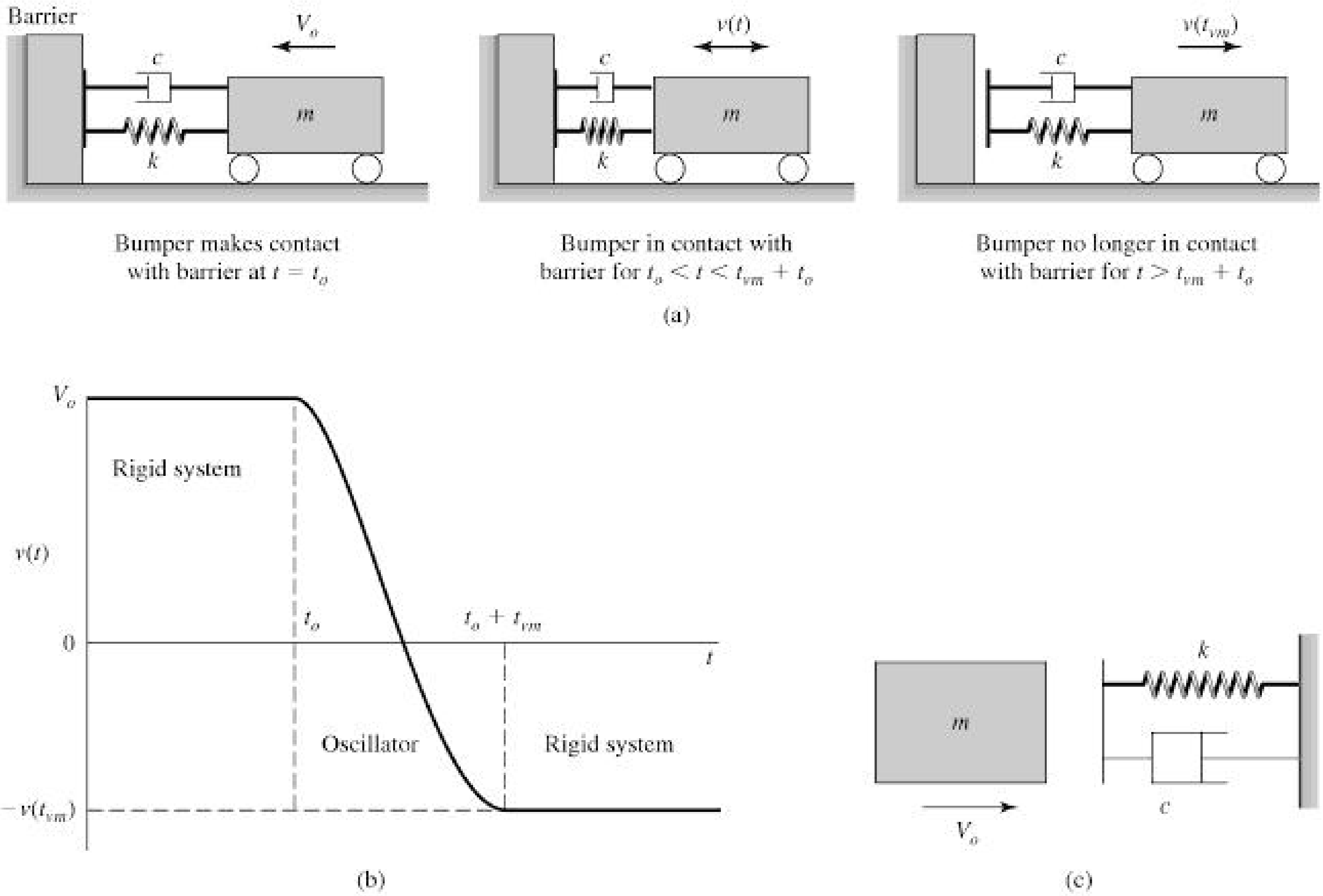

FIGURE 4.6

(a) Model of a car bumper colliding with a stationary barrier, (b) time history of velocity of mass, and (c) equivalent impact configuration. In this equivalent configuration, a mass moving with a velocity V_o impacts a barrier, which is represented by a spring and damper combination.

The amount of energy that the system dissipates during the interval $0 \le t \le t_{vm}$ is the difference between the initial kinetic energy and the kinetic energy at separation. Note that the vehicle does not have any potential energy when it is not in contact with the barrier. Thus,

$$E_{\text{diss}} = \frac{1}{2}mV_o^2 - \frac{1}{2}m[\dot{x}(t_{vm})]^2 = \frac{1}{2}mV_o^2 - \frac{1}{2}mV_o^2 e^{-4\varphi/\tan\varphi}$$

or

$$\frac{E_{\text{diss}}}{E_{\text{init}}} = [1 - e^{-4\varphi/\tan\varphi}]$$

where E_{init} is given by Eq. (4.27). These results are summarized in Figure 4.7. It is noted that these results have been obtained for a collision with a single impact.

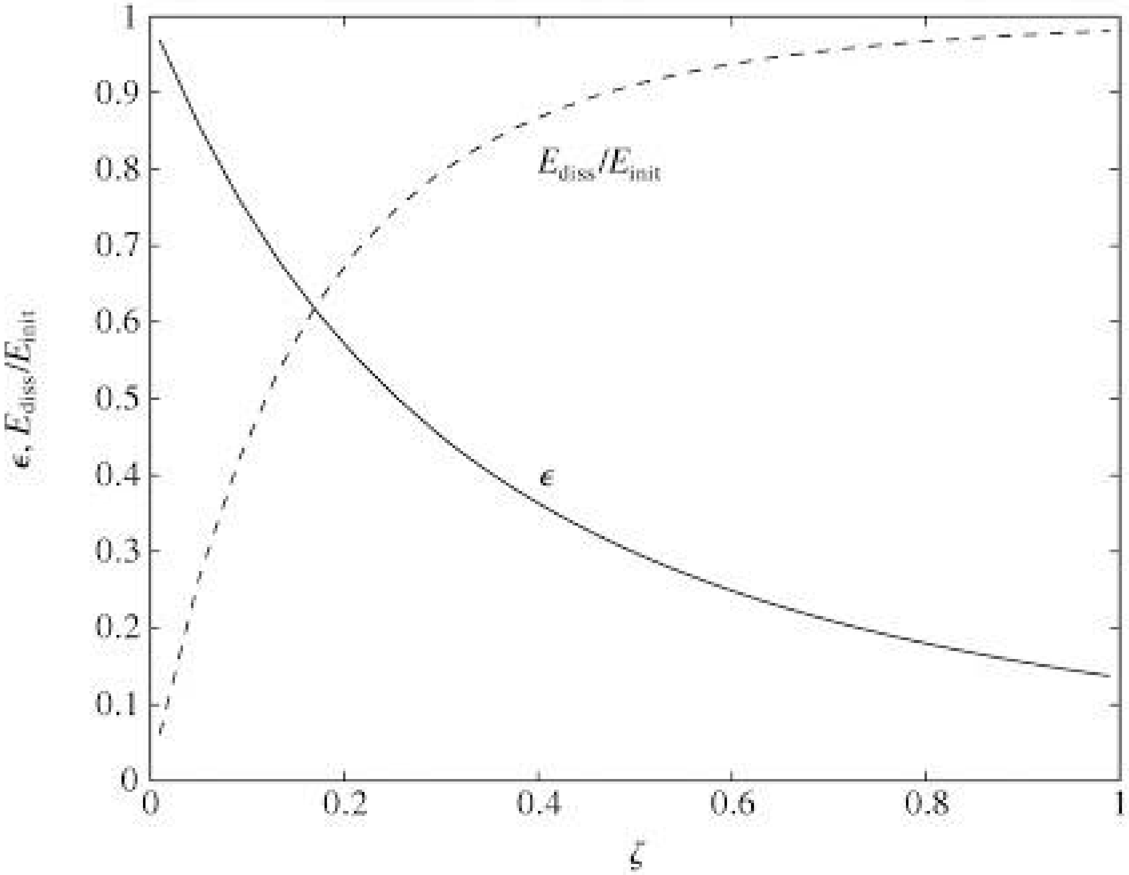

FIGURE 4.7
Coefficient of restitution and fraction of energy dissipation for impacting single degree-of-freedom system.

EXAMPLE 4.5 Impact of a container housing a single degree-of-freedom system

We shall now consider the effects of dropping onto the floor a system that resides inside a container that has a coefficient of restitution ϵ with respect to the floor. The system is shown in Figure 4.8. If the container falls from a height h, then the magnitude of the velocity at the time of impact with the floor is

$$V_o = \sqrt{2gh}$$

At the instant $t = 0^+$ after impact, the container bounces upwards with a velocity whose magnitude is ϵV_o. Then at $t = 0^+$, the container and the single degree-of freedom system can be modeled as a single degree-of freedom system with a moving base as discussed in Section 3.5. Thus, if we define the relative displacement

$$z(t) = x(t) - y(t) \tag{a}$$

then, from Eq. (3.30) we have

$$m\frac{d^2z}{dt^2} + c\frac{dz}{dt} + kz = -m\frac{d^2y}{dt^2} \tag{b}$$

However, $\ddot{y} = -g$, since the container is decelerating during the rebound upwards. Then Eq. (b) becomes

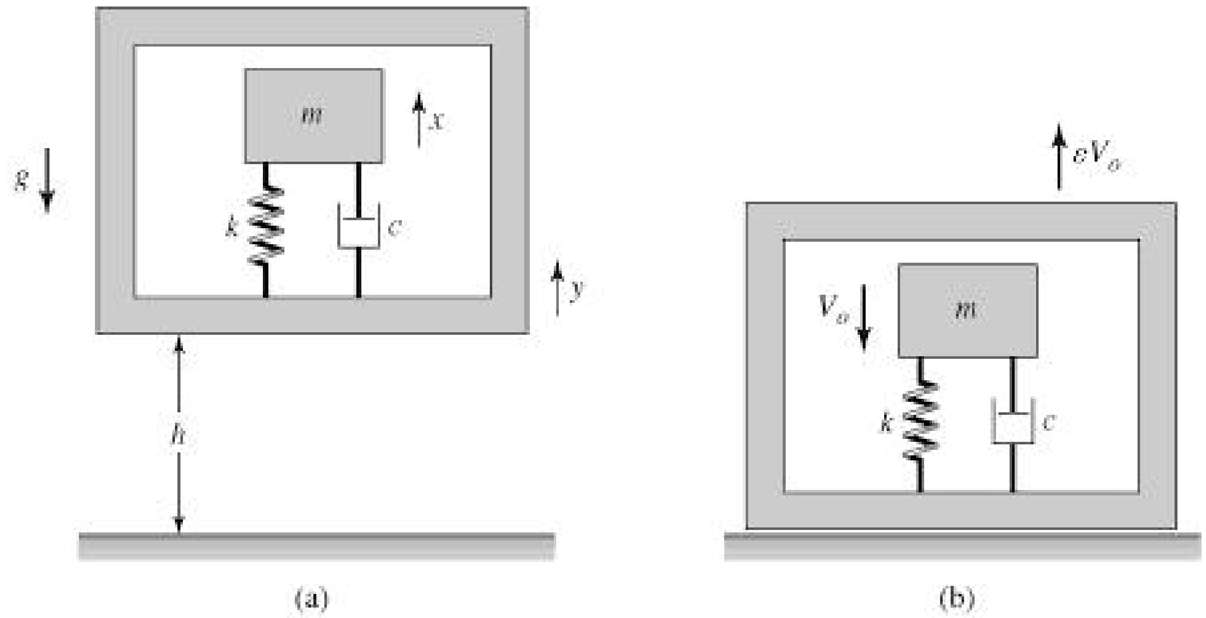

FIGURE 4.8
Single degree-of-freedom system inside a container: (a) dropped from a height h and (b) on rebound immediately after impact with the floor.

$$m\frac{d^2z}{dt^2} + c\frac{dz}{dt} + kz = mgu(t) \tag{c}$$

where $u(t)$ is the unit step function. The initial conditions are

$$\begin{aligned} z(0) &= x(0) - y(0) = 0 \\ \dot{z}(0) &= \dot{x}(0) - \dot{y}(0) = -V_o - (\epsilon V_o) \\ &= -(1+\epsilon)V_o = -(1+\epsilon)\sqrt{2gh} \end{aligned} \tag{d}$$

The solution to Eq. (c) for $0 < \zeta < 1$ and subject to the initial conditions given by Eq. (d) is determined from Eq. (D.11). Thus, after substituting $f(t) = mgu(t)$, we find that

$$\begin{aligned} \frac{z(t)}{\delta_{st}} &= \frac{-R}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\,\omega_n t) \\ &\quad + 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\,\omega_n t + \varphi) \end{aligned} \tag{e}$$

where φ is given by Eq. (4.15), $\delta_{st} = mg/k$, and the coefficient of restitution-dependent parameter R is

$$R = (1+\epsilon)\sqrt{\frac{2h}{\delta_{st}}} \tag{f}$$

The corresponding velocity is

$$\begin{aligned} \frac{\dot{z}(t)}{\delta_{st}\omega_n} &= \frac{R}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\,\omega_n t - \varphi) \\ &\quad + \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\,\omega_n t) \end{aligned} \tag{g}$$

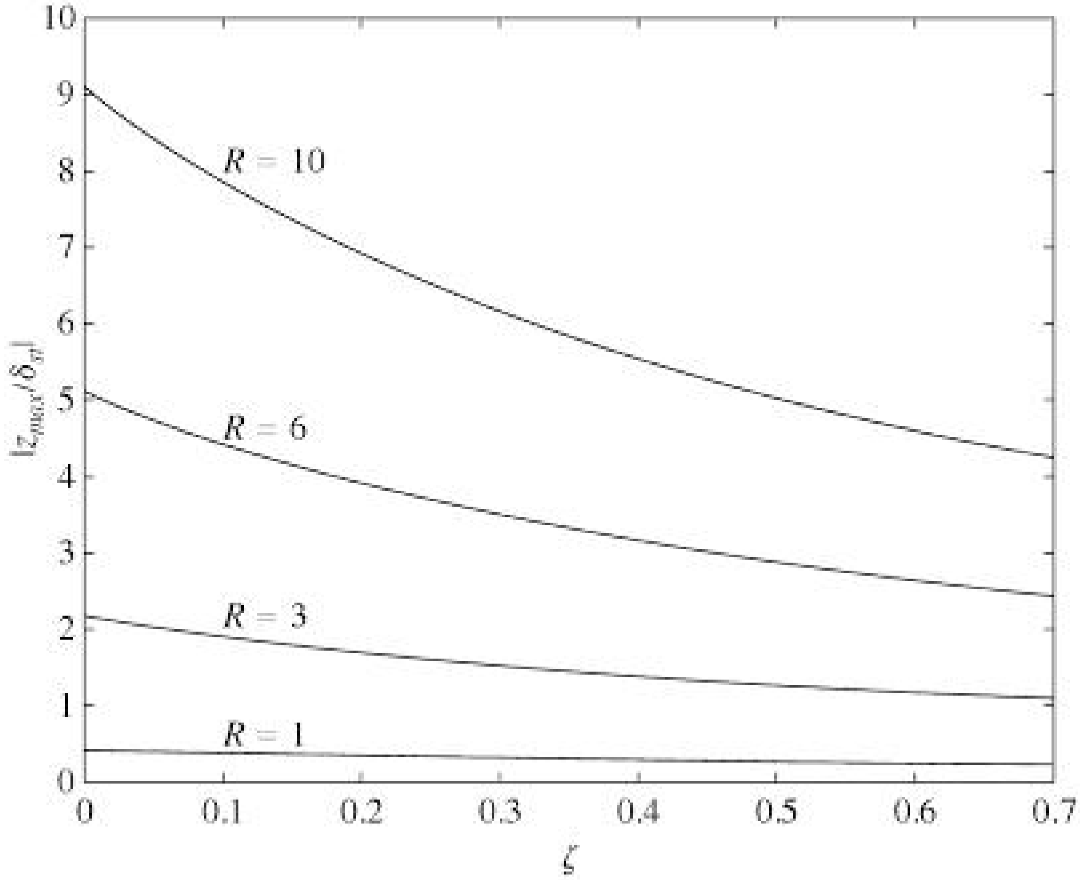

FIGURE 4.9

Normalized maximum relative displacement of a system inside a container that is dropped from a height h as a function of coefficient of restitution of the container and the damping ratio of the single degree-of-freedom system.

The extremum of the relative displacement is determined from $z_{\max} = z(t_{\max})$ where $t_{\max}$ is the earliest time at which $\dot{z}(t_{\max}) = 0$. In this particular case, an explicit analytical expression for $t_{\max}$ cannot be found, so the maximum displacement is determined numerically from Eq. (e). The magnitude of the maximum displacement is a function of the initial velocity, which is a function of the drop height h, the coefficient of restitution of the container, and the damping ratio and the static displacement of the spring of the single degree-of-freedom system inside the container. The numerically obtained results are shown in Figure 4.9. We see that there are many ways by which one can decrease the maximum relative displacement of the mass, which lead to the following design guidelines.

Design Guidelines: To minimize the maximum relative displacement of the mass, keep h and, therefore the velocity V_o, as small as possible. Make the container of a material that absorbs the impact, so that the coefficient of restitution is as small as possible. Make the natural frequency of the single degree-of-freedom system as low as possible. Since, in packaging, the mass is usually not a parameter that can be specified, one has to make the equivalent spring as soft as practical. Increase the equivalent damping of the packing material.

EXAMPLE 4.7 Vibratory system employing a Maxwell model

We shall now modify the single degree-of-freedom system shown in Figure 3.1 to obtain a more realistic description of the reaction force transmitted to the fixed support, when the inertial element is given an initial velocity. As noted earlier, if a Kelvin-Voigt model is used, there is an instantaneous reaction force at the base when an initial velocity is imparted to the mass. This unrealistic response is eliminated by using the modified system shown in Figure 4.11a, where we have introduced a linear spring k_1 in series with a linear viscous damper c. The combination of the linear spring k_1 in series with the linear viscous damper c is called a *Maxwell model.* To describe the motion of the system, we need, in addition to the displacement variable x of the mass m, another displacement variable x_d to describe the displacement at the spring-damper junction in the Maxwell model. Both x and x_d are measured from their respective static-equilibrium positions.

Governing Equations of Motion and Solution for Response

The governing equations are obtained for the general case with forcing and, from this case, the free response of the mass subjected to an initial velocity is

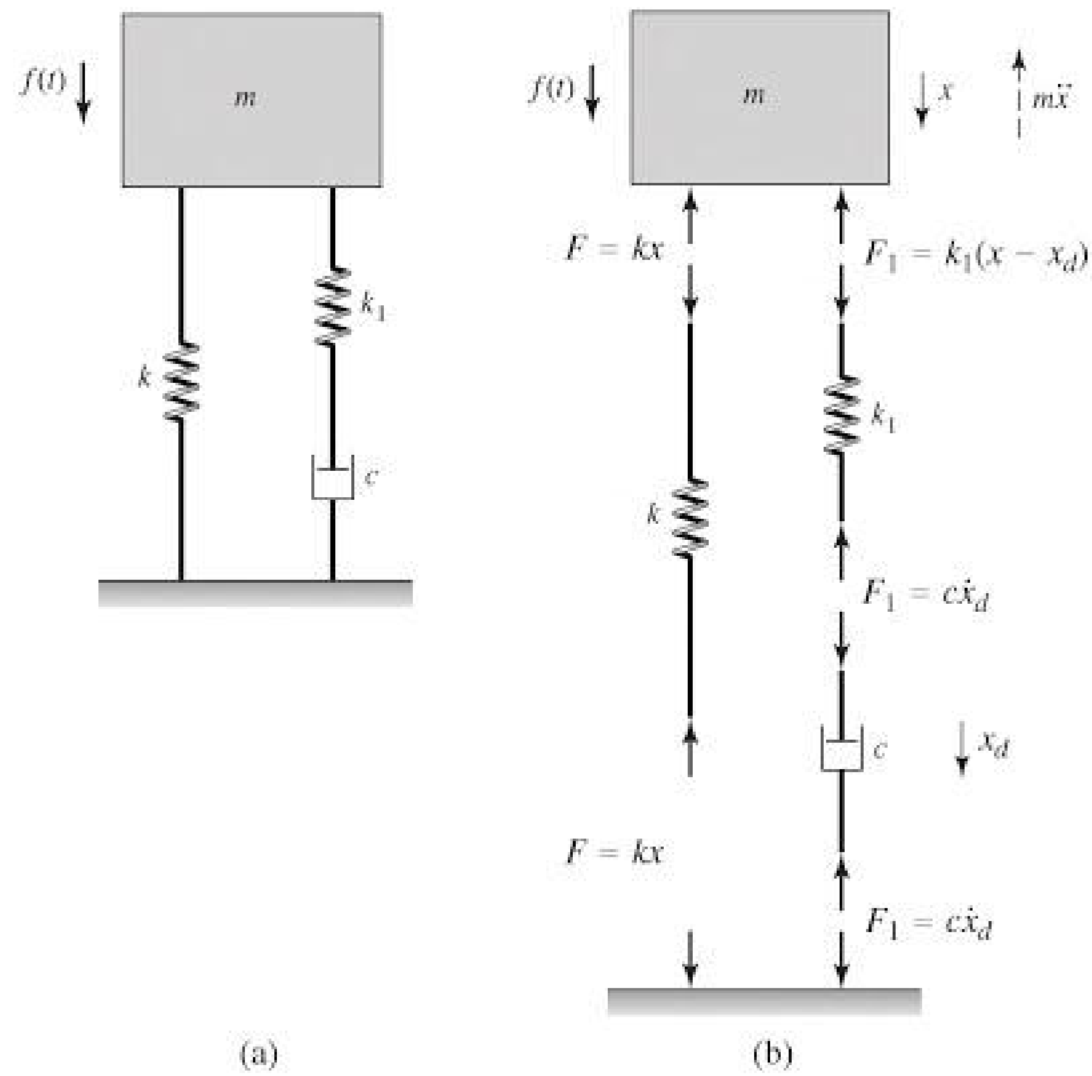

FIGURE 4.11
(a) Single degree-of-freedom system with a spring added in series with the damper and (b) forces on the system's elements.

determined. Making use of Figure 4.11b and carrying out a force balance along the vertical direction for the mass m and a force balance for the Maxwell element, we arrive at

$$m\frac{d^2x}{dt^2} + kx + k_1(x - x_d) = f(t)$$
$$k_1(x - x_d) = c\frac{dx_d}{dt} \tag{a}$$

Since we have an additional first-order equation, apart from the second-order equation typical of a single degree-of-freedom system, the vibratory system of Figure 4.11 is also referred to as a *one and a half degree-of-freedom system.*

Introducing the natural frequency

$$\omega_n = \sqrt{\frac{k}{m}} \tag{b}$$

and the nondimensional quantities

$$\tau = \omega_n t$$
$$\gamma = \frac{k_1}{k} \tag{c}$$

Eqs. (a) are rewritten as

$$\ddot{x} + (1 + \gamma)x - \gamma x_d = f(\tau)/k$$
$$\gamma x - \gamma x_d = 2\zeta\dot{x}_d \tag{d}$$

and the overdot indicates the derivative with respect to τ and

$$2\zeta = \frac{c\omega_n}{k} \tag{e}$$

In the limiting case, when $\gamma \to \infty$ (i.e., $k_1 \to \infty$), the second of Eqs. (d) leads to a Kelvin-Voigt model with a linear spring of stiffness k in parallel with a linear damper with damping coefficient c. Therefore, Eqs. (d) can be used to study a vibratory system with a Maxwell model as well as a Kevin-Voigt model.

If we represent the Laplace transform of $x(\tau)$ by $X(s)$, the Laplace transform of $x_d(\tau)$ by $X_d(s)$, and the Laplace transform of $f(\tau)$ by $F(s)$, then, from pair 2 in Table A of Appendix A, the Laplace transforms of Eqs. (d) are

$$(s^2 + 1 + \gamma)X(s) - \gamma X_d(s) = G(s)$$
$$\gamma X(s) - (\gamma + 2\zeta s)X_d(s) = 0 \tag{f}$$

where we have assumed that $x_d(0) = 0$, and used the notation

$$G(s) = \frac{F(s)}{k} + sx(0) + \dot{x}(0) \tag{g}$$

Upon solving for $X(s)$ and $X_d(s)$ from Eqs. (f), we obtain, respectively,

$$X(s) = \frac{G(s)(\gamma + 2\zeta s)}{2\zeta s^3 + \gamma s^2 + 2\zeta(1 + \gamma)s + \gamma}$$

$$X_d(s) = \frac{\gamma G(s)}{2\zeta s^3 + \gamma s^2 + 2\zeta(1 + \gamma)s + \gamma} \tag{h}$$

Force Transmitted to the Fixed Support

From Figure 4.11b, the reaction force on the base is seen to be

$$F_B = F_1 + F = c\frac{dx_d}{dt} + kx \tag{i}$$

which, in terms of the nondimensional quantities given by Eqs. (c), is written as

$$\frac{F_B}{k} = 2\zeta \dot{x}_d + x \tag{j}$$

where the overdot is the derivative with respect to τ. Upon taking the Laplace transform of Eq. (j), again assuming that $x_d(0) = 0$, and using Eqs. (h), we find that

$$\frac{F_B}{k} = \frac{G(s)[\gamma + 2\zeta(1 + \gamma)s]}{2\zeta s^3 + \gamma s^2 + 2\zeta(1 + \gamma)s + \gamma} \tag{k}$$

This expression will be revisited in Example 5.13.

We shall limit the rest of our discussion to the case where the applied force and the initial displacement are zero; that is, $f(\tau) = 0$ and $x(0) = 0$, and the initial velocity is

$$\frac{dx(0)}{dt} = \omega_n \frac{dx(0)}{d\tau} = V_o \tag{l}$$

Therefore, Eq. (g) simplifies to

$$G(s) = \frac{V_o}{\omega_n} \tag{m}$$

Upon substituting Eq. (m) into Eq. (k), we arrive at

$$\frac{F_B}{(kV_o/\omega_n)} = \frac{\gamma + 2\zeta(1 + \gamma)s}{2\zeta s^3 + \gamma s^2 + 2\zeta(1 + \gamma)s + \gamma} \tag{n}$$

Before evaluating Eq. (n), we recall that the limiting case when $\gamma \to \infty$ (i.e., $k_1 \to \infty$) recovers the Kelvin-Voigt model, where a linear spring k is in parallel with a linear viscous damper c. For this limiting case, we divide the numerator and denominator of Eq. (n) by γ and take the limit as $\gamma \to \infty$. This operation results in

$$\frac{F_B}{(kV_o/\omega_n)} = \frac{1 + 2\zeta s}{s^2 + 2\zeta s + 1} \tag{o}$$

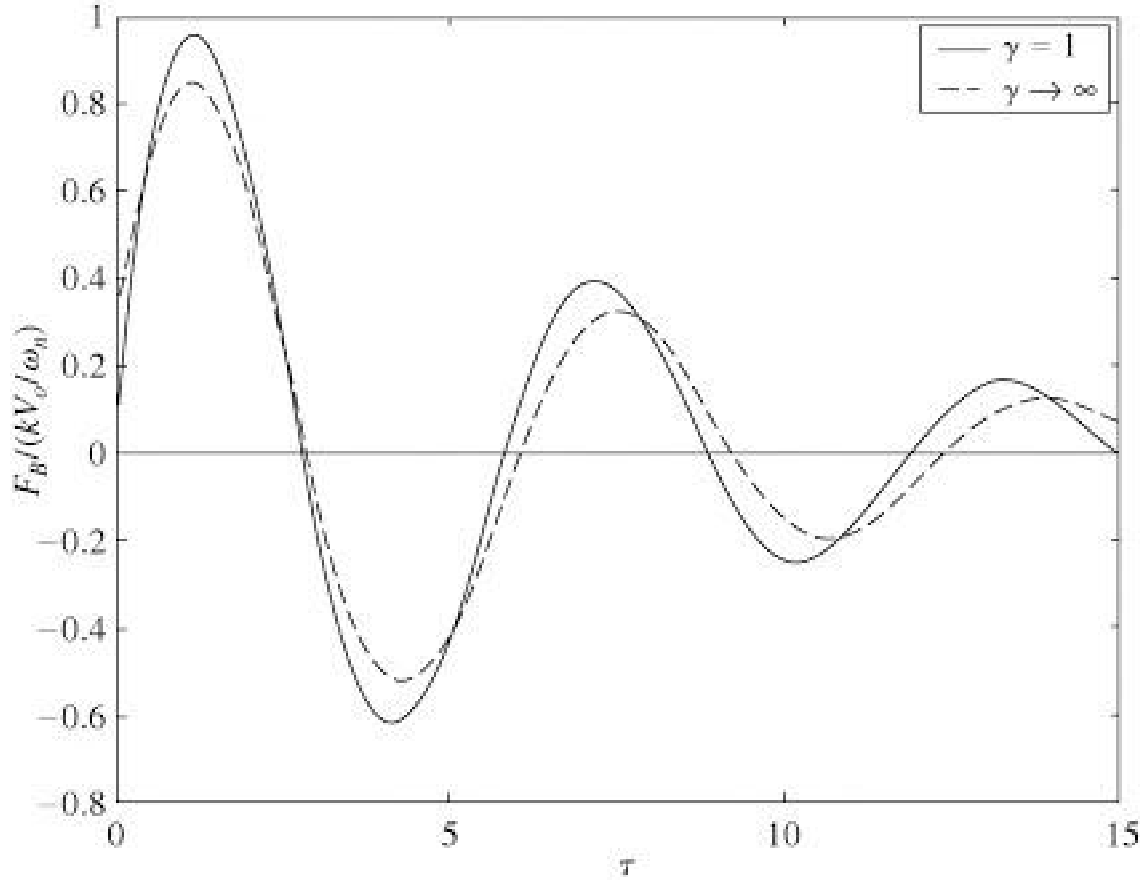

FIGURE 4.12
Reaction force of the system shown in Figure 4.11 for $\zeta = 0.15$.

Upon using Laplace transform pairs 14 and 16 in Table A of Appendix A, the inverse Laplace transform of Eq. (o) results in Eq. (4.23). The numerically[4] computed inverse Laplace transforms of Eq. (n) for $\zeta = 0.15$ and $\gamma = 1$ and Eq. (o) for $\zeta = 0.15$ are shown in Figure 4.12. At $\tau = 0$, we see that the reaction force F_B has a discontinuity for the Kevin-Voigt model, while this reaction force is zero for the Maxwell model.

EXAMPLE 4.8 Vibratory system with Maxwell model revisited

As a continuation of Example 4.7, we now consider the case where the support consists only of a Maxwell element; that is, the spring k is absent. In this case, we again examine the force transmitted to the fixed base. Setting $k = 0$ in Eq. (a) of Example 4.7, we arrive at

$$m\frac{d^2x}{dt^2} + k_1(x - x_d) = f(t)$$

$$k_1(x - x_d) = c\frac{dx_d}{dt} \tag{a}$$

[4]The MATLAB function `ilaplace` from the Symbolic Toolbox was used.

Introducing a new set of quantities

$$\tau' = \omega_{1n} t \qquad \text{and} \qquad \omega_{1n}^2 = \frac{k_1}{m} \tag{b}$$

Eq. (a) is written as

$$\begin{aligned} \ddot{x} + x - x_d &= f(t)/k_1 \\ x - x_d &= 2\zeta_1 \dot{x}_d \end{aligned} \tag{c}$$

where the overdot now indicates the derivative with respect to τ' and

$$2\zeta_1 = \frac{c\omega_{1n}}{k_1} \tag{d}$$

If we represent the Laplace transform of $x(\tau')$ by $X(s)$, the Laplace transform of $x_d(\tau')$ by $X_d(s)$, and the Laplace transform of $f(\tau')$ by $F(s)$, then, from pair 2 in Table A of Appendix A, the Laplace transforms of Eqs. (c) are

$$\begin{aligned} (s^2 + 1)X(s) - X_d(s) &= G_1(s) \\ X(s) - (1 + 2\zeta_1 s)X_d(s) &= 0 \end{aligned} \tag{e}$$

where we have assumed that $x_d(0) = 0$ and

$$G_1(s) = \frac{F(s)}{k_1} + sx(0) + \dot{x}(0) \tag{f}$$

Upon solving for $X(s)$ and $X_d(s)$ in Eqs. (e), we obtain

$$\begin{aligned} X(s) &= \frac{G_1(s)(1 + 2\zeta_1 s)}{s(2\zeta_1 s^2 + s + 2\zeta_1)} = \frac{G_1(s)(2\zeta_m + s)}{s(s^2 + 2\zeta_m s + 1)} \\ X_d(s) &= \frac{G_1(s)}{s(2\zeta_1 s^2 + s + 2\zeta_1)} \end{aligned} \tag{g}$$

where

$$2\zeta_m = \frac{1}{2\zeta_1} = \frac{k_1}{c\omega_{1n}} \tag{h}$$

Note that $\zeta_m < 1$ only when $\zeta_1 > 0.25$.

When the spring with stiffness k is absent, the reaction force on the base is

$$F_B(t) = F_1 = c\frac{dx_d}{dt} \tag{i}$$

which is rewritten in terms of the nondimensional quantity given by Eq. (b) as

$$\frac{F_B(\tau')}{k_1} = 2\zeta_1 \dot{x}_d \tag{j}$$

4.2.3 Initial Displacement

We now examine the free response of an underdamped single degree-of-freedom system with a prescribed initial displacement. When a system is subjected to an initial displacement only, we set $V_o = 0$ and Eq. (4.6) for the amplitude and phase simplify to

$$A_o = \frac{X_o}{\sqrt{1-\zeta^2}}$$
$$\varphi_d = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \varphi$$

Therefore, Eq. (4.4), which describes the displacement response, becomes

$$x(t) = \frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi) \tag{4.28}$$

and, after using Eq. (D.12), the velocity and acceleration are, respectively,

$$\dot{x}(t) = v(t) = -\frac{X_o\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t)$$
$$\ddot{x}(t) = a(t) = \frac{X_o\omega_n^2}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t - \varphi) \tag{4.29}$$

Equations (4.28) and (4.29) are plotted in Figure 4.14 and the corresponding state space plot is shown in Figure 4.15 along with their respective time histories. As time unfolds, the trajectory is attracted to the equilibrium position located at the origin (0, 0).

Logarithmic Decrement[6]

Consider the displacement response of a single degree-of-freedom system subjected to an initial displacement as shown in Figure 4.16. The logarithmic decrement δ is defined as the natural logarithm of the ratio of any two successive amplitudes of the response that occur a period T_d apart, where T_d is given by

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{4.30}$$

From these two amplitudes, it is possible to determine the damping ratio ζ. To this end, we determine a relationship between the logarithmic decrement and the damping factor. We start from

$$\delta = \ln\left(\frac{x(t)}{x(t+T_d)}\right) \tag{4.31}$$

[6]Although the definition of the logarithmic decrement is provided in Section 4.2.3, it applies equally to all free responses considered in Section 4.2.

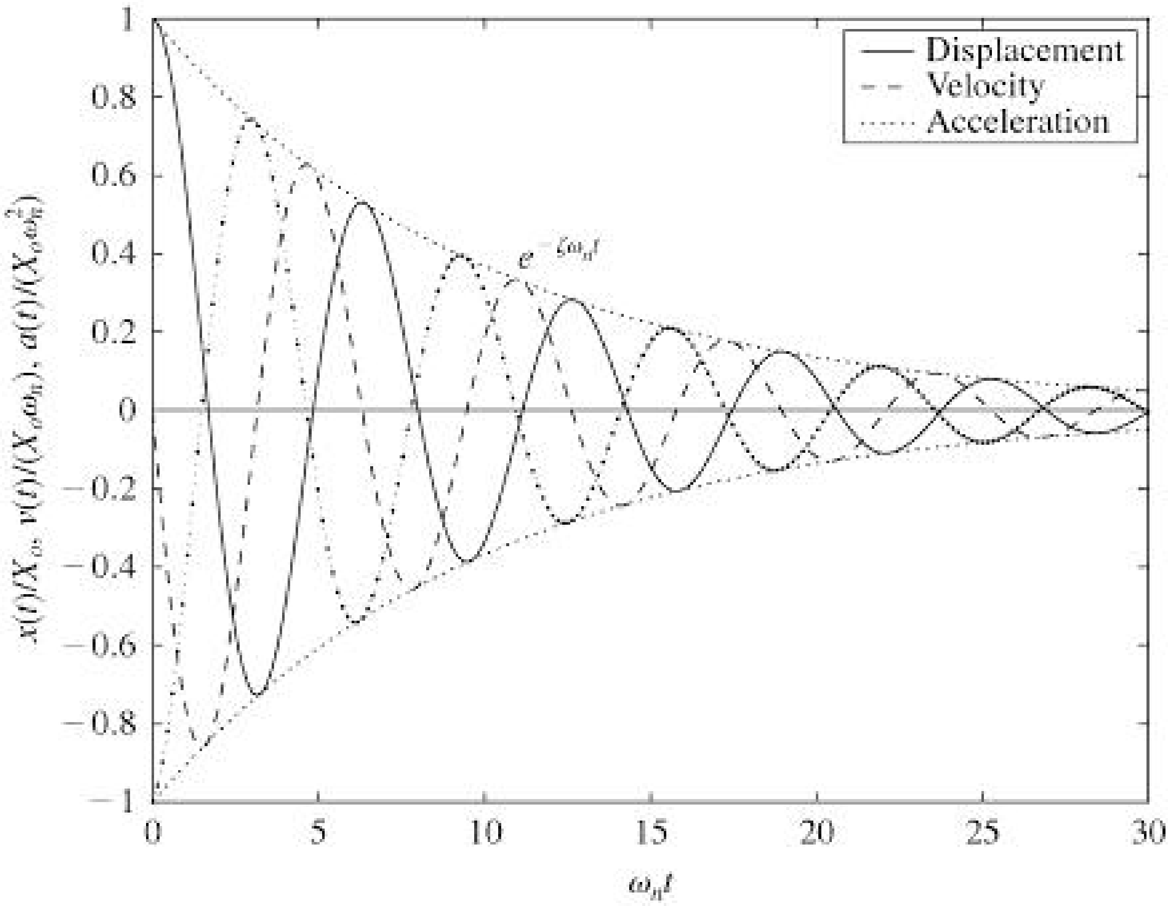

FIGURE 4.14

Time histories of displacement, velocity, and acceleration of a system with a prescribed initial displacement.

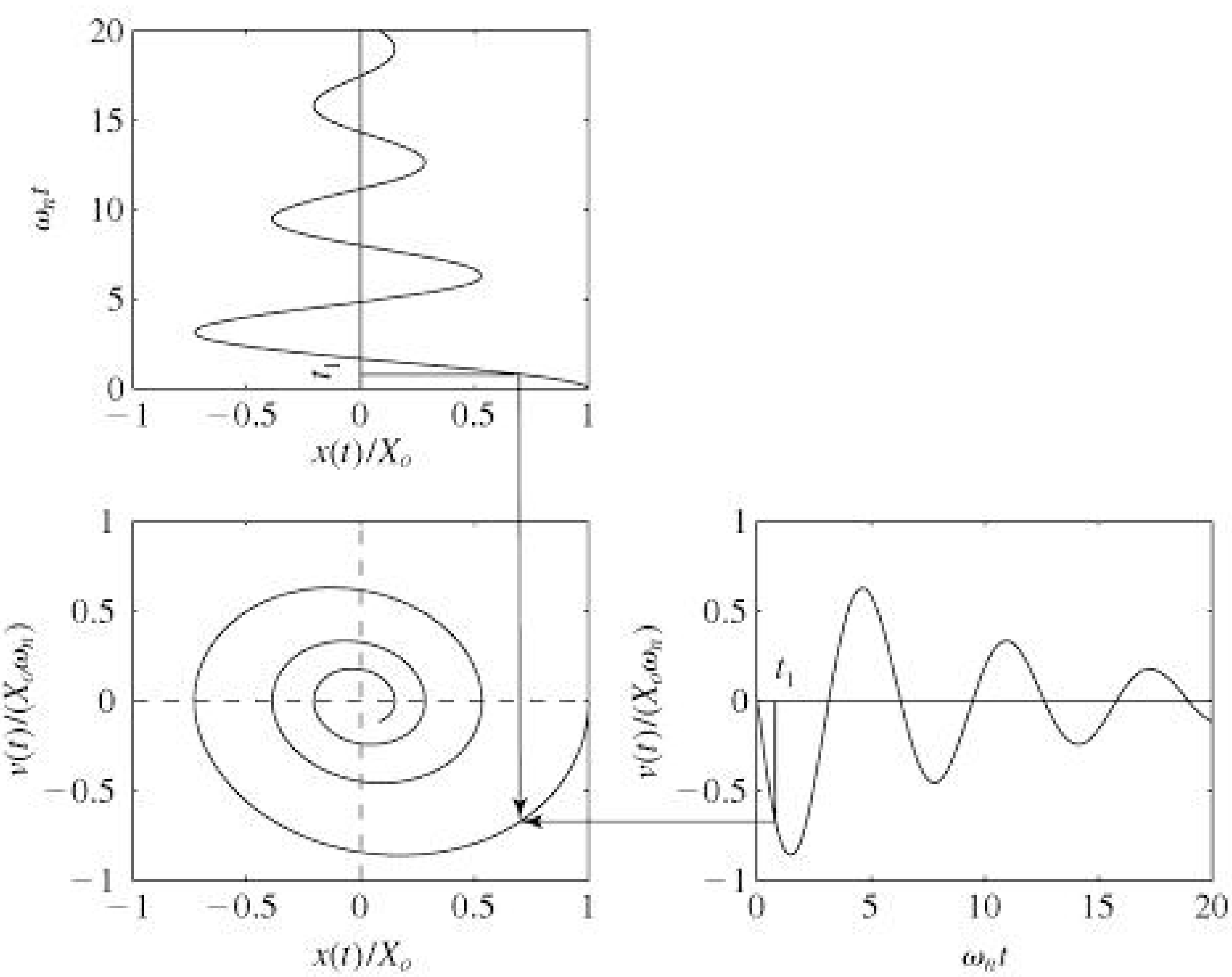

FIGURE 4.15

State-space plot of single degree-of-freedom system with prescribed initial displacement.

squares represent the data through which the fitted curve is depicted as a solid line. As discussed in Section 5.3.2, the estimation of parameters such as damping factor and natural frequency can be also carried out based on the system transfer function.

EXAMPLE 4.9 Estimate of damping ratio using the logarithmic decrement

It is found from a plot of the response of a single degree-of-freedom system to an initial displacement that at time t_o the amplitude is 40% of its initial value. Two periods later the amplitude is 10% of its initial value. We shall determine an estimate of the damping ratio. Thus, from Eq. (4.34)

$$\delta = \frac{1}{2}\ln\left(\frac{0.4}{0.1}\right) = 0.693$$

Then, from Eq. (4.36), we find that

$$\zeta = \frac{1}{\sqrt{1 + (2\pi/0.693)^2}} = 0.11$$

4.2.4 Initial Displacement and Initial Velocity

We shall now consider the case when a system is subjected to an initial displacement and an initial velocity simultaneously. The solution is given by Eq. (4.4), which is repeated below for convenience.

$$x(t) = A_o e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi_d) \tag{4.37}$$

From Eqs. (4.6), we find that the amplitude and phase are given by

$$\begin{aligned} A_o &= \sqrt{X_o^2 + \left(\frac{V_o + \zeta\omega_n X_o}{\omega_d}\right)^2} = X_o\sqrt{1 + \frac{(V_r + \zeta)^2}{1 - \zeta^2}} \\ \varphi_d &= \tan^{-1}\frac{\omega_d X_o}{V_o + \zeta\omega_n X_o} = \tan^{-1}\frac{\sqrt{1 - \zeta^2}}{\zeta + V_r} \end{aligned} \tag{4.38}$$

and $V_r = V_o/(\omega_n X_o)$ is a velocity ratio. The velocity response is determined from Eq. (4.37) to be

$$\dot{x}(t) = -A_o\omega_n e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi_d - \varphi) \tag{4.39}$$

where φ is given by Eq. (4.15).

The numerically evaluated result for $x(t)/X_o$ is shown in Figure 4.18. For "small" values of V_r, the displacement response is similar to that obtained for a system with a prescribed initial displacement and for "large" values of V_r, the displacement response is similar to that obtained for a system with a prescribed initial velocity.

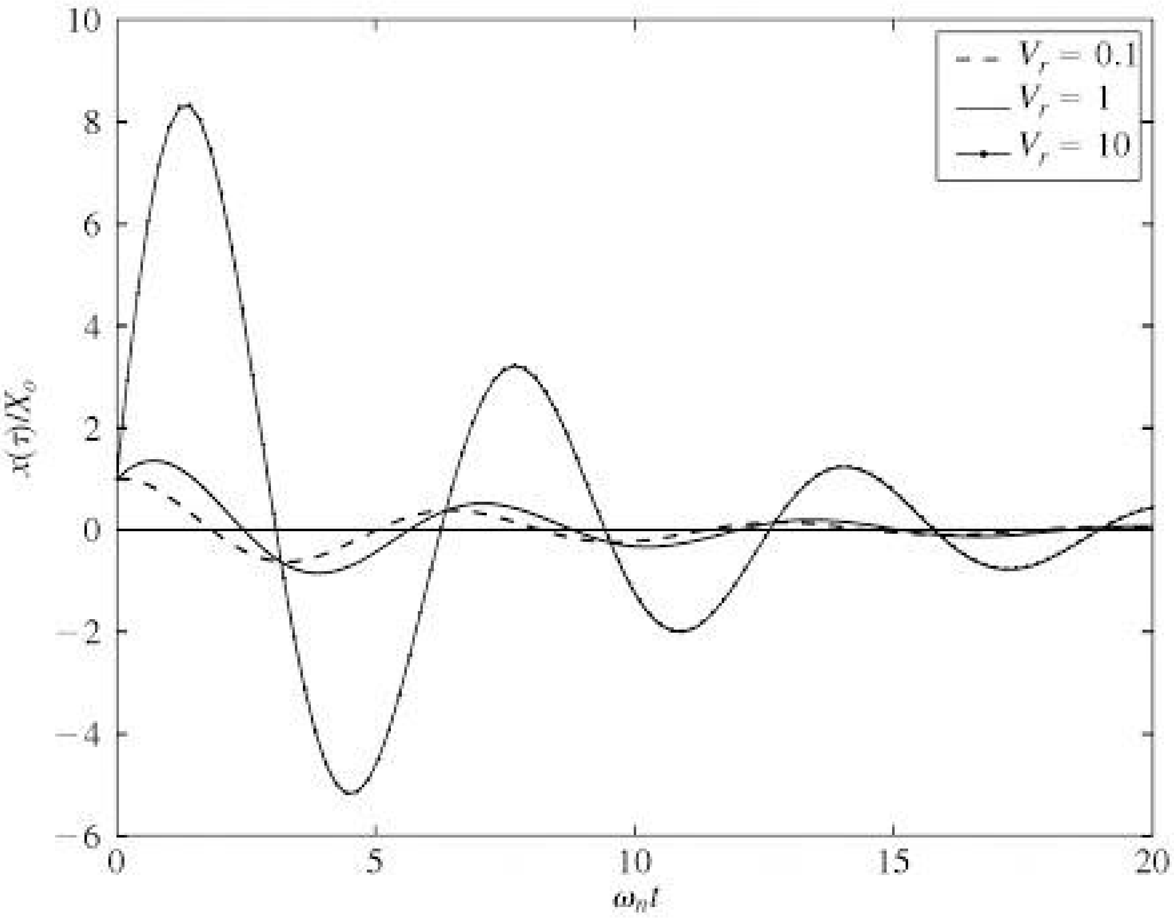

FIGURE 4.18
Displacement response of a system with prescribed initial displacement and prescribed initial velocity.

EXAMPLE 4.10 Inverse problem: information from a state-space plot

Consider the state-space plot shown in Figure 4.19. From this graph, we shall determine the following: (a) the value of the damping ratio and (b) the time $\tau_{max} = \omega_n t_{max}$ at which the maximum displacement occurs.

From the graph, the initial conditions are $x(0) = X_o$ and $v(0) = 1.6X_o\omega_n$. To determine ζ, the logarithmic decrement is used. For convenience, we select the values of the displacement from Figure 4.19 that are along the line $v(t) = 0$. Then,

$$x(t) \approx 0.95X_o \qquad \text{and} \qquad x(t + T_d) \approx 0.5\,X_o \tag{a}$$

and from Eq. (4.34) and Eq. (a), we determine the logarithmic decrement

$$\delta = \ln\left(\frac{0.95\,X_o}{0.5X_o}\right) = \ln(1.90) = 0.642 \tag{b}$$

Then, from Eq. (4.36) and Eq. (b), we find the damping factor to be

$$\zeta = \frac{1}{\sqrt{1 + (2\pi/\delta)^2}} = \frac{1}{\sqrt{1 + (2\pi/0.642)^2}} = 0.10 \tag{c}$$

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4.4 MACHINE TOOL CHATTER

In Figure 4.21, a model of a turning operation on a lathe is shown. When the cutting parameters such as spindle speed and width of cut are carefully chosen, the turning operation can produce the desired surface finish on the work piece. However, this turning operation can become unstable for certain values of spindle speed and width of cut. When these undesirable conditions are present, the tool and work piece system "chatters," producing an undesirable surface finish and a shortening of tool life. In this section, we shall explore the loss of stability that leads to the onset of chatter.

For a rigid work piece and a flexible tool, the cutting force acting on the tool due to the uncut material and the associated damping can be modeled as shown in Figure 4.21. The mass m represents the mass of the tool and tool holder, k is the stiffness of the tool holder's support structure, and c is the equivalent viscous damping of the structure. The dynamic cutting force F_c is the sum of the forces due to the change in chip thickness and the change in the penetration rate of the tool.[11] Thus, we have

$$F_c = \underbrace{k_1}_{\text{Cutting stiffness}} \underbrace{[x(t) - \mu x(t - 2\pi/N)]}_{\text{Change in chip thickness}} + \underbrace{K\frac{2\pi}{N}}_{\text{Damping}}\frac{dx}{dt}$$

where μ is the overlap factor ($0 \le \mu \le 1$), k_1 is an experimentally determined dynamic coefficient called the *cutting stiffness*, K is the experimentally determined penetration rate coefficient, and N is the rotational speed of either the tool or the work piece in revolutions per second. Then carrying out a force balance based on Figure 4.21, the tool vibrations can be described by the following equation

$$\frac{d^2x}{d\tau^2} + \left(\frac{1}{Q} + \frac{K}{k\Omega}\right)\frac{dx}{d\tau} + \left(1 + \frac{k_1}{k}\right)x - \underbrace{\mu\frac{k_1}{k}x(\tau - 1/\Omega)}_{\text{Time-delay effect due to uncut chip during previous pass}} = 0 \quad (4.52a)$$

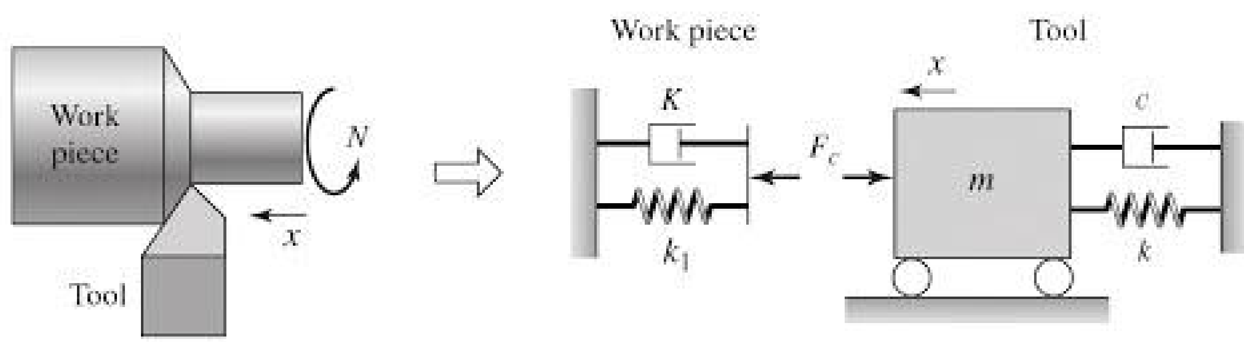

FIGURE 4.21
Model of a tool and work piece during turning.

[11] S. A. Tobias, *Machine-Tool Vibration,* Blackie & Sons, Ltd., Glasgow, pp. 146–176 (1965).

In Eqs. (4.54), the quantities K/k, μ, and k_1/k are known, and the values of the nondimensional spindle speed Ω are varied over a specified range. At each value of Ω, the value of ω is determined numerically[12] from the second of Eqs. (4.54). The values for Ω and ω are then used in the first of Eqs. (4.54) to determine the positive values of Q that satisfy the equation; that is, those values of Ω and ω for which

$$\frac{1}{Q} = -\frac{K}{k\Omega} - \frac{\mu k_1}{k}\frac{\sin(\omega/\Omega)}{\omega}$$

In the plot of Ω versus Q, we can show the regions for which the system is either stable or unstable. Representative results are shown in Figure 4.22. The shaded regions, which are in the form of lobes, are regions of instability, and are referred to as stability lobes. The asymptote to these lobes is shown in the figure by a dashed line. If one conservatively chooses the cutting parameters so that one is below this asymptote to the lobes, then based on the linear theory presented here, the tool will not chatter. Of course, one can also choose spindle speeds that correspond to regions between the stability lobes as well.

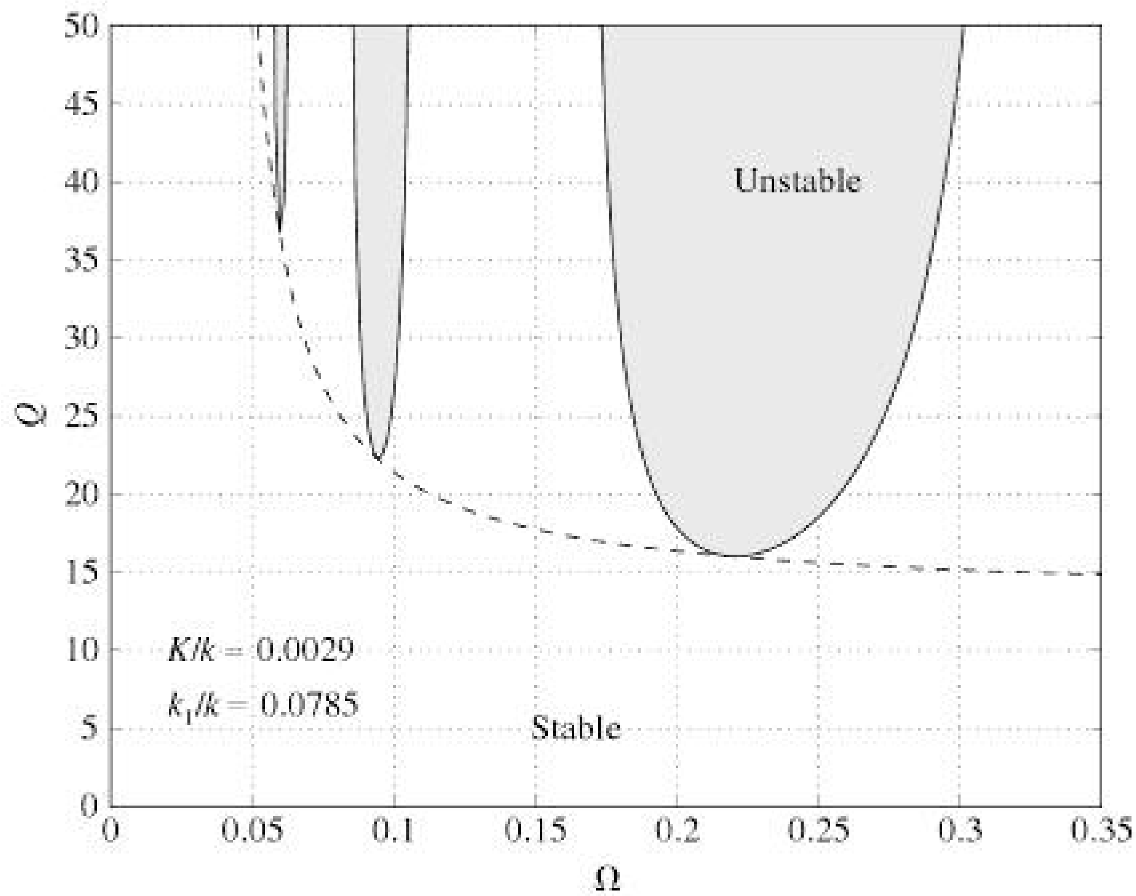

FIGURE 4.22
Stability chart for one set of parameters in turning: $\mu = 1$.

[12]The MATLAB function `fzero` was used.

4.5 SINGLE DEGREE-OF-FREEDOM SYSTEMS WITH NONLINEAR ELEMENTS

4.5.1 Nonlinear Stiffness

We illustrate the effects that two different types of nonlinear springs can have on the free response of a system when subjected to either an initial displacement or initial velocity.

System with Hardening Cubic Spring

First, we consider a system that has a spring whose stiffness includes a component that varies as the cube of the displacement. After using Eq. (2.23) for the nonlinear spring force, the governing equation is

$$\frac{d^2x}{d\tau^2} + 2\zeta\frac{dx}{d\tau} + x + \alpha x^3 = 0 \tag{4.55}$$

where the nondimensional time variable $\tau = \omega_n t$. We solve Eq. (4.55) numerically,[13] since it does not have an analytical solution. We assume that $\alpha = 1\ \text{cm}^{-2}$, $\zeta = 0.15$, and that the initial conditions are $X_o = 2$ cm and $V_o = 0$. The results are shown in Figures 4.23, along with the solution for the system with a linear spring; that is, when $\alpha = 0$.

We see from these results that the response of the system with the nonlinear spring is distinctly different from that with the linear spring. First, the response of the system with the nonlinear spring does not decay exponentially with time and second, the displacement response does not have a constant period of damped oscillation. These differences provide one a means of distinguishing one type of nonlinear system from a linear system based on an examination of the response to an initial displacement. In practice, the nonuniformity of the period is easier to detect, since the dependence of frequency (or period) on the amplitude of free oscillation is a characteristic of a nonlinear system.

System with Piecewise Linear Springs

We now consider a second nonlinear system shown in Figure 4.24. In this case, the springs are linear; however, the mass is straddled by two additional linear elastic spring-stops that are not contacted until the mass has been displaced by an amount d in either direction. The stiffness of the springs is proportional to the attached spring by a constant of proportionality μ ($\mu \geq 0$). When $\mu = 0$, we have the standard linear single degree-of-freedom system, and when $\mu > 1$, the elastic spring-stops are stiffer than the spring that is permanently attached to the mass. The governing equation describing the motion of the system is[14]

$$\frac{d^2y}{d\tau^2} + 2\zeta\frac{dy}{d\tau} + y + \mu h(y) = 0$$

[13] The MATLAB function ode45 was used.

[14] H. Y. Hu, "Primary Resonance of a Harmonically Forced Oscillator with a Pair of Symmetric Set-up Elastic Stops," *J. Sound Vibration*, Vol. 207, No. 3, pp. 393–401, 1997.

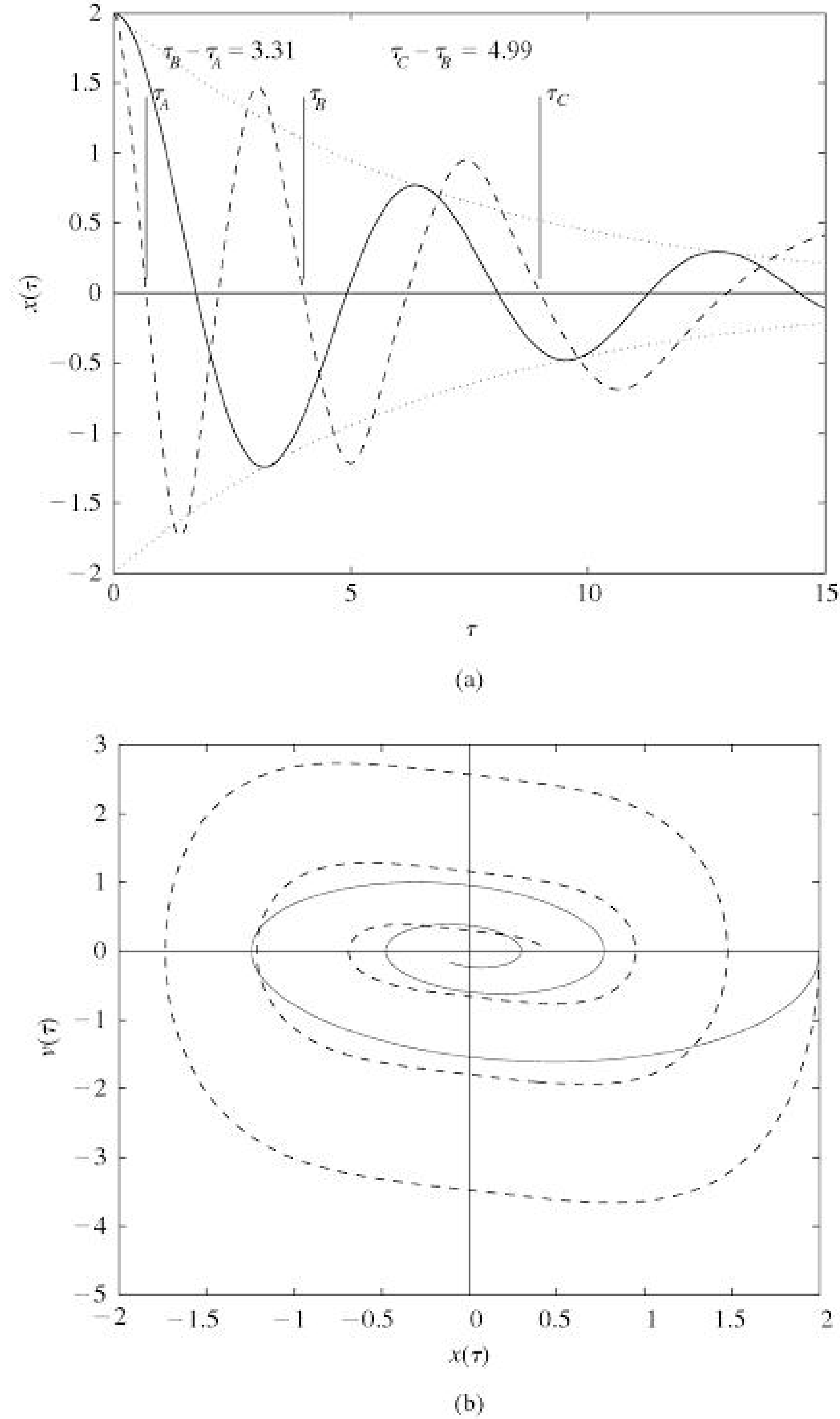

FIGURE 4.23

Comparison of the responses of linear (solid lines) and nonlinear (dashed lines) systems with prescribed initial displacement: (a) displacement and (b) phase portrait.

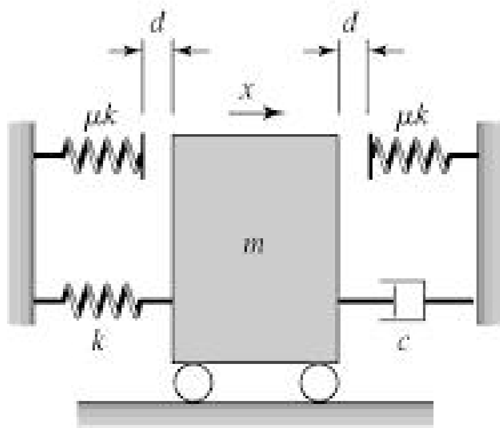

FIGURE 4.24
Single degree-of-freedom system with additional springs that are not contacted until the mass displaces a distance d in either direction.
Source: Reprinted by permission of Federation of the European Biochemical Societies from Journal of Sound Vibration, 207, H.Y.Hu., FEBS Letters, "Primary Resonance of a Harmonically Forced Oscillator with a Pair of Symmetric Set-Up Elastic Stops," pp.393–401, Copyright © 1997, with permission from Elsevier Science.

where

$$h(y) = 0 \qquad |y| \le 1$$
$$= y - \operatorname{sgn}(y) \qquad |y| > 1$$

and, as discussed in Section 2.4.2, the signum function sgn(y) is +1 when $y > 0$ and is −1 when $y < 0$. Furthermore, we have employed the following definitions:

$$\tau = \omega_n t, \quad \omega_n = \sqrt{\frac{k}{m}}, \quad y = \sqrt{\frac{x}{d}}, \quad \text{and} \quad 2\zeta = \frac{c}{m\omega_n}$$

Although it is possible to find a solution for this piecewise linear system, here we obtain a numerical[15] solution for convenience. We shall determine the response of this system when it is subjected to an initial (dimensionless) velocity $dy(0)/d\tau = V_o/(\omega_n d) = 10$, the damping factor $\zeta = 0.15$, and the values of μ are 0, 1, and 10. The results are shown in Figure 4.25. We see that the introduction of the spring-stops decreases the amplitude of the mass. In addition, it has the effect of decreasing the period of oscillation, which is equivalent to increasing its natural frequency.

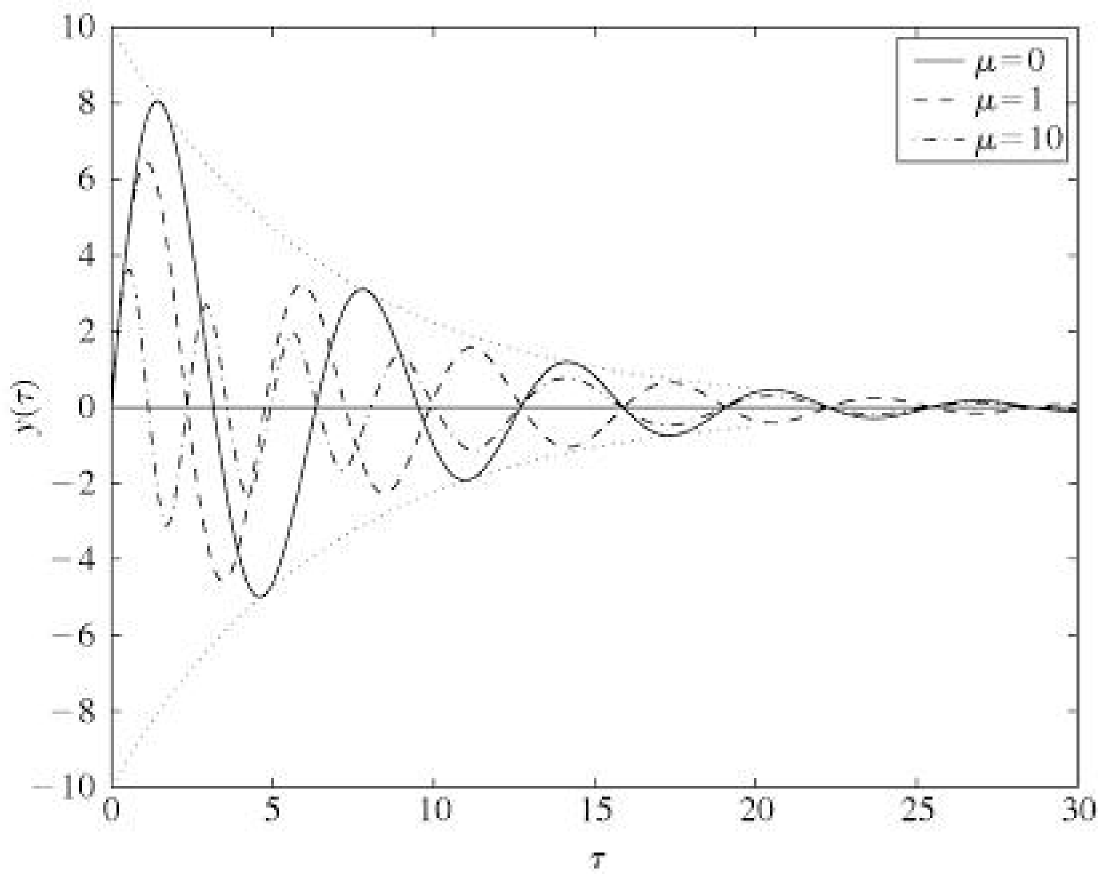

FIGURE 4.25
Response of the system shown in Figure 4.24 with prescribed initial velocity $V_o/(\omega_n d) = 10$.

[15]The MATLAB function ode45 was used.

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